【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. (10 points) Use Gauss-Jordan reduction to solve the system below.

$$\begin{cases} 2x+y+z+w=-1 \\ x-2y-3z-4w=-14 \\ 2y+z+w=5 \\ 2x-2z-w=-1 \end{cases}$$

2. (10 points) Use Cramer's Rule to solve z in the following system.

$$\begin{cases} 2x+5y-2z+7w=3 \\ x-2y-5z-4w=-10 \\ 3y+4z+w=8 \\ x-3z+2w=-3 \end{cases}$$

3. Let $B=\{\mathbf{b}_1=(2,2,3)^T, \mathbf{b}_2=(-1,5,2)^T, \mathbf{b}_3=(1,-2,2)^T\}$ and $\mathbf{v}=(20,-22,19)^T$.

   A. (3 points) Find $2\mathbf{b}_1+3\mathbf{b}_2+4\mathbf{b}_3$.

   B. (5 points) Find the coordinate vector $[\mathbf{v}]_B$.

   C. (7 points) Find the coordinate transition matrix from the standard basis to the basis B.

4. (5 points) Is the set $\{(1,2,3)^T, (-1,1,0)^T, (3,-2,1)^T\}$ a basis of $\mathbf{R}^3$? Prove your answer.

5. (10 points) Let $\mathbf{A}=\begin{bmatrix} 2 & 3 & -2 & 7 \\ 5 & 2 & -2 & 5 \\ 1 & -4 & 2 & -9 \end{bmatrix}$. Find $\text{rank}(\mathbf{A})$, $\text{nullity}(\mathbf{A})$, and $\dim(\text{column space of } \mathbf{A})$.

6. (5 points) Write the rotation matrix in R^3 which rotates an angle θ counterclockwise about the y-axis.

7. (10 points) Let $L: V \to W$ be a linear transformation.

   Prove or disprove the following two statements:

   (1) If $v_1, v_2, \ldots, v_k$ are linearly dependent in V, then $L(v_1), L(v_2), \ldots, L(v_k)$ are linearly dependent in W.

   (2) If $v_1, v_2, \ldots, v_k$ are linearly independent in V, then $L(v_1), L(v_2), \ldots, L(v_k)$ are linearly independent in W.

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8. (10 points) Consider the vector space $C[0, 2\pi]$ with inner product

$$<f, g> = \int_0^{2\pi} f(x)g(x)dx.$$

Let subspace $S = span\{1, \cos x, \sin x, \cos 2x, \sin 2x\}$.

(1) Find the norm of each vector in the basis.

(2) $f(x) \in C[0, 2\pi]$, find a function in S which is the best least squares approximation of $f(x)$ with respect to the given inner product. Express your answer in terms of integration.

9. (10 points) In R^3, find the orthogonal projection of $(1,0,0)^T$ onto the subspace S, where $S = \text{span}\{(1,1,0)^T, (1,1,1)^T\}$.

10. (15 points)

(1)

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$ Find a matrix B such that $B^2 = A$

(2)Let T be a linear transformation $T: R^2 \to R^2$ given by

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 3x-2y \\ -x+2y \end{bmatrix}$$

Compute $T^{100}(\begin{bmatrix} x \\ y \end{bmatrix}) = ?$

Where the notation $T^{100}(\begin{bmatrix} x \\ y \end{bmatrix})$ is defined recursively as follows

$$T^1(\begin{bmatrix} x \\ y \end{bmatrix}) = T(\begin{bmatrix} x \\ y \end{bmatrix}) \qquad T^n(\begin{bmatrix} x \\ y \end{bmatrix}) = T^{n-1}(T(\begin{bmatrix} x \\ y \end{bmatrix})).$$

(3) Prove that if A is diagonalizable with n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\det A = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$.