東海大學105學年度碩士班考試入學試題

計算證明題（需有完整計算過程，否則不予計分）：　**請於答案卷作答，違者不予計分**

1. (24%=8%+8%+8%) Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

   (a) Find $\det(A)$.

   (b) Find A^{-1}.

   (c) Solve the system of linear equations

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

2. (24%=8%+8%+8%) 判斷下列各小題的向量爲 linearly independent 或 linearly dependent (須有完整過程)

   (a) $\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

   (b) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ in $\mathbb{R}^{2\times 2}$.

   (c) $\{x, x+1, x^2+1\}$ in $P_3 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$.

3. (10%) Let $E = \{\mathbf{u_1}, \mathbf{u_2}\}$ and $F = \{\mathbf{v_1}, \mathbf{v_2}\}$ be ordered basis for $\mathbb{R}^2$, where $\mathbf{u_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and $\mathbf{v_1} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$. Determine the transition matrix (轉換矩陣) corresponding to a change of basis from $\{\mathbf{u_1}, \mathbf{u_2}\}$ to $\{\mathbf{v_1}, \mathbf{v_2}\}$.

4. (10%) Let $A = \begin{bmatrix} 1 & 3 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 2 & 2 \\ 3 & 4 & 3 & 3 & 3 \end{bmatrix}$. Find $rank(A)$ and $nullity(A)$.

5. (18%=6%+6%+6%) True or false: 以下敘述，先回答它是對(T)或錯(F)的，對的話請寫出證明，錯的話請舉出反例。

(a) $S = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 + x_2 = x_3 \}$ 是向量空間$\mathbb{R}^3$ 的一個子空間(subspace)。

(b) 對任意的 2×2非奇異(nonsingular)矩陣A，$det(A^{-1}) = \frac{1}{det(A)}$。

(c) 若A, B均爲$n \times n$矩陣且存在一個非$\mathbf{0}$向量$\mathbf{x}_0 \in \mathbb{R}^n$使得 $A\mathbf{x}_0 = B\mathbf{x}_0$，則$A - B$必爲奇異矩陣(singular matrix)。

6. (14%) Find the **eigenvalues** and the corresponding **eigenvectors** for the matrix

$$A = \begin{bmatrix} 4 & 1 & 0 \\ -5 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

and then find $\mathbf{A^4}$.