1. (20%)
The probability density function (pdf) and the characteristic function of a Cauchy random variable X are given as $f_X(x) = \frac{a}{\pi(x^2+a^2)}$, $-\infty < x < \infty$ and $\Phi_X(\omega) = e^{-a|\omega|}$, respectively. Now let $X_1, X_2, \cdots, X_n$ be n independent Cauchy random variables with identical pdf $f_X(x)$. Define $Y_n = \frac{1}{n}(X_1 + \cdots + X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i$.
(a) Find the characteristic function of Y_n.
(b) Find the pdf of Y_n.
(c) Find $\lim_{n\to\infty} \Phi_{Y_n}(\omega)$ using part (a).
(d) Will $\lim_{n\to\infty} \Phi_{Y_n}(\omega)$ approach the characteristic function of a Gaussian random variable? Please explain.

2. (10%)
Ordering a "deluxe" cake means you have three choices from 10 available toppings. Assuming that the order in which the toppings are selected does not matter.
(a) How many combinations are possible if the toppings can be repeated?
(b) How many combinations are possible if the toppings cannot be repeated?

3. (10%)
Random variables X and Y have a joint probability mass function (PMF) given by the following matrix:

$P_{X,Y}(x,y)$	$y=-1$	$y=0$	$y=1$
$x=-1$	0	0.25	0
$x=1$	0.25	0.25	0.25

(a) Are X and Y independent? Please explain your answer.
(b) Are X and Y uncorrelated? Please explain your answer.

4. (10%)
Let X be a continuous random variable with probability density function (pdf) :
$f_X(x) = e^{-\pi x^2}$, $-\infty < x < \infty$.
(a) Find the expected value of X.
(b) Find the variance of X.

5. There are three recursive functions

$$x_n = x_{n-1} + 0.5\, y_{n-1}$$
$$y_n = z_{n-1} + 0.5\, y_{n-1}$$
$$z_n = 0$$

where $\boldsymbol{n}$ is a positive integer. These relationships can be hold in the condition

$$x_n + y_n + z_n = 1 \ (\boldsymbol{n} \text{ is a non-negative integer}).$$

Let $\boldsymbol{A}^{(n)} = [x_n\, y_n\, z_n]^T$ be the n-th power of the matrix $\boldsymbol{A}$.

(a) (5 %) Write these recursive equations in power form of matrix $\boldsymbol{A}$ with a coefficient matrix $\boldsymbol{C}$.

(b) (5 %) Find the rank of $\boldsymbol{C}$.

(c) (15 %) Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{C}$.

(d) (5%) Calculate the values of x_n, y_n, and z_n with the skill of "Diagonalization" as n approaches infinity.

6. If $p=\sum_{i=0}^{4} a_i x^i$ and $q=\sum_{i=0}^{4} b_i x^i$ are any vectors in $\boldsymbol{P}_4$, then $\langle p,q\rangle=\sum_{i=0}^{4} a_i b_i$ is an inner product in $\boldsymbol{P}_4$. Let V be the subspace of $\boldsymbol{P}_4$ spanned by the vectors

$$p_1 = 2+2x-x^2+x^4,$$
$$p_2 = -1-x+2x^2-3x^3+x^4,$$
$$p_3 = 1+x-2x^2-x^4,$$
$$p_4 = x^2+x^3+x^4.$$

(a) (5 %) Find the orthogonal complement of V.

(b) (5 %) Find a subset of $\{p_i |\ i = 1,2,3,4\}$ to be a basis of V.

7. Let $M=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 4 & 4 \\ 1 & 2 & -2 & -3 \end{bmatrix}$

(a) (5 %) Show the determinant of M.

(b) (5 %) Calculate

$$\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & 4 & 4 \\ 1 & 2 & -2 & -3 \\ 4 & 4 & 4 & 4 \end{bmatrix}+\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ -2 & -2 & 3 & 3 \\ 2 & 3 & -1 & -2 \end{bmatrix}.$$