1. (15%) Let $F(x, y) \in \Re^2$ be an electric force field in a planar region, which is given by

$$F(x, y) = \begin{bmatrix} F_1(x, y) \\ F_2(x, y) \end{bmatrix} = \begin{bmatrix} 4x + 3y \\ x + 2y \end{bmatrix}$$

We want to move a particle along a straight line from point $P_1(1, 2)$ to point $P_2(5, 3)$, denoted by L.

(a) (5%) Please explain the physical meaning of the following line integral

$$\int_L F_1(x, y)dx + F_2(x, y)dy$$

(b) (10%) Please provide a procedure for computing the above line integral. **Note:** You are NOT required to do the calculation. Only a procedure is needed.

2. (10%) Consider the partial differential equation given by

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial x}$$

where $0 \le x \le 1$ and $0 \le t < \infty$. The boundary conditions are

$$u(0, t) = u(1, t) = 0$$

and the initial condition is

$$u(x, 0) = x(1 - x)$$

Please provide a procedure for solving the partial differential equation for the function $u(x, t)$. **Note:** You do NOT need to actually find $u(x, t)$. Only a procedure is needed.

3. (25%) Please find the solution of $\boldsymbol{x}(t)$, given $\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}(t)$ with $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ and $\boldsymbol{x}(0) = [5, -7]^T$

4. (12%) Solve the differential equation using Laplace transform.

$y'' + 2y' - 3y = 8e^{-t} + \delta(t - \frac{1}{2}),\ y(0) = 3,\ y'(0) = -5$

5. (8%) Solve the integral equation by convolution.

$y(t) = \sin 2t + \int_0^t y(\tau) \sin 2(t - \tau) d\tau$

6. (5%) Given $F(s) = \mathcal{L}[f(t)]$, find $f(t)$.

$\frac{3s - 5}{s^2 - 6s + 13}$

7. (25%)

(a). Show that the vertical motion of the mechanical system in the following figure (no damping, masses of springs neglected) is governed by the simultaneous differential equations (5%)

$$m_1\ddot{y}_1 = -k_1y_1 + k_2(y_2 - y_1)$$

$$m_2\ddot{y}_2 = -k_2(y_2 - y_1)$$

where dots denote derivatives with respect to the time, t.

(b). The system of equations in prob. (a) may be written as a single vector equation $\ddot{\mathbf{y}} = \mathbf{Ay}$. Please determine matrix $\mathbf{A}$. (5%)

(c). To solve the equation in prob. (b) $\ddot{\mathbf{y}} = \mathbf{Ay}$, substitute $\mathbf{y} = \mathbf{x}e^{\omega t}$ and show that this leads to the eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{x}$, and determine the expression of λ. (5%)

(d). Let $m_1 = m_2 = 1, k_1 = 2, k_2 = 3$, find the eigenvalues and corresponding eigenvectors. (5%)

(e). Find the general solution of $\mathbf{y}(t)$. (5%)