1. (10%) Determine whether the vectors
   $\mathbf{v}_1 = (1, 2, 2, -1)$, $\mathbf{v}_2 = (4, 9, 9, -4)$, $\mathbf{v}_3 = (5, 8, 9, -5)$
   are linearly independent. Please justify your answer.

2. (10%) Calculate the scalar triple product $\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w})$ of the vectors
   $\mathbf{u} = (5, 1, 0)$, $\mathbf{v} = (6, 2, 0)$, $\mathbf{w} = (4, 2, 2)$.

3. (10%) Find the values of k for which A is non-invertible.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ k & 2 & k \\ 2 & 4 & 2 \end{bmatrix}$$

4. (10%) Find a matrix P that diagonalizes A and compute A^{100}, where

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

5. (10%) Let the vector space P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

   Apply the Gram-Schmidt process to transform the standard basis $\{1, x, x^2\}$ for P_2 into an orthogonal basis $\{\phi_1(x), \phi_2(x), \phi_3(x)\}$.

6. Determine the truth value of each of these statements if the universe of discourse of each variable consists of all real numbers.
   a) (3%) $\forall x \exists y (x^2 = y)$
   b) (3%) $\forall x \exists y (x = y^2)$
   c) (3%) $\exists x \forall y (xy = 0)$
   d) (3%) $\forall x \exists y (x + y = 1)$
   e) (3%) $\exists x \exists y (x + 2y = 2 \wedge 2x + 4y = 5)$

7. Prove or disprove each of these statements about the floor and ceiling functions.
   a) (5%) $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$ for all real numbers *x* and *y*.
   b) (5%) $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ for all real numbers *x* and *y*.

8. (10%) How many r-digits binary sequences that have no adjacent 1s are there? Justify your answer.

9. In this problem, we consider only undirected graphs without self-loops. Let G be a

graph with two connected components. The numbers of vertices of the connected components are p and q, respectively. We assume that p>q>2. Answer the following questions (No explanations are necessary).

a) (3%) How many vertices do we need to choose to ensure that two of them are from different components?
b) (3%) What is the possible maximum number of edges in G?
c) (3%) What is the possible minimum number of edges in G?
d) (3%) If we randomly pick two vertices, what is the probability that the two vertices are in different components?
e) (3%) Let H be the transitive closure of G. If we randomly pick three vertices in H, what is the probability that the three vertices form a triangle (clique) in H?