1. (35%) Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 \le x \le 1, 0 < \theta < \infty, \\ 0 & \text{elsewhere} \end{cases}$$

(i) (7%) Find the maximum likelihood estimator (MLE) of θ. Is the MLE of θ unbiased? Show the details.

(ii) (4%) Find the Rao-Cramer's lower bound of MLE of θ.

(iii) (3%) Let W be Beta(1,1), and consider $Y = -\theta \log W$, where $\theta > 0$. Find the cdf of Y.

(iv) (6%) Suppose the random sample $Y_1, \ldots, Y_n$ from the distribution of Y. Find the MLE and the minimum variance unbiased estimator (MVUE) of $P(Y \le 2)$).

(v) (5%) Suppose the random sample $Y_1, \ldots, Y_n$ from the distribution of Y. Find a uniformly most powerful critical region of size $\alpha = 0.05$ and $n = 2$ for testing $H_0 : \theta = 2$ against $H_a : \theta > 2$. Show the details. ($\chi^2_{(2)} = 5.991, \chi^2_{(3)} = 7.815, \chi^2_{(4)} = 9.488, \chi^2_{(5)} = 11.071, \chi^2_{(6)} = 12.592$)

(vi) (4%) Let the random sample $W_1, \ldots, W_n$ from Beta(1,1) and let $W_{(1)}, \ldots, W_{(n)}$ denote the order statistics. Find the distribution of the j^{th} order statistic $W_{(j)}$. Show the details.

(vii) (6%) Let the random sample $W_1, \ldots, W_n$ from Beta(1,1) and let $W_{(1)}, \ldots, W_{(n)}$ denote the order statistics. The range is defined as $R = W_{(n)} - W_{(1)}$ and the mid-range is defined as $V = (W_{(1)} + W_{(n)})/2$. Find the distribution of R.

2. (8%) Consider X and Y have a trinomial distribution with joint pmf

$$p(x,y) = \frac{n!}{x!y!(n-x-y)!} p_1^x p_2^y p_3^{n-x-y},$$

where x and y are nonnegative integers with $x+y \le n$;
$p_1,\ p_2,\ p_3 \in (0,1)$ and $p_1+p_2+p_3=1$; and let $p(x,y)=0$ elsewhere.

   (i) (4%) Find the moment generating function of a trinomial distribution.

   (ii) (4%) Compute $E(Y|X=x)$.

3. (13%) Let $X \sim N(0,\ \theta)$ where $0<\theta<\infty$.

   (i) (3%) Show that the family $N(0,\ \theta)$ where $0<\theta<\infty$ is not complete by finding at least one nonzero function $u(x)$ such that E(u(X))=0, for all $\theta > 0$.

   (ii) (6%) If $X_1,\ X_2, \cdots,\ X_n$ be a random sample from $N(0,\ \theta)$, show that the MLE of θ is an efficient estimate of θ.

   (iii) (4%) Find the minimum variance unbiased estimator of θ^2.

4. (20%) For the simple linear regression model, $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, where $i=1,\ldots,n$ and ϵ_i are i.i.d $N(0,\sigma^2)$.

   (i) (4%) To derive the least squared estimator(s) (LSE) for β_0 and β_1.

   (ii) (5%) Find the sampling distribution of $\hat{\beta}_1$. Show the details.

   (iii) (5%) For testing $H_0 : \beta_1 = 0$. Find the distribution of $SSReg/\sigma^2$ under H_0 is true, where $SSReg$ is the sum of squares regression.

   (iv) (6%) Derive the likelihood ratio test for testing $H_0 : \beta_1 = 0$ against $H_a : \beta_1 \neq 0$.

5. (24%) Let $X_1, \ldots, X_n$ be a random sample from a distribution, where $Pr(X_i = 1) = \theta$, $Pr(X_i = 0) = 1 - \theta$, where $0 < \theta < 1$.

(i) (5%) Show that this is a uniformly most powerful test when we test $H_0 : \theta = \frac{1}{2}$ against $H_a : \theta < \frac{1}{2}$.

(ii) (5%) Based on (i). Use the Central Limit Theorem (CLT) to find n and c so that the significance level is approximately 0.10 and the power of the test is approximately 0.80 when $\theta = \frac{1}{3}$.

(iii) (6%) If the prior p.d.f of Θ is

$$f(\theta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}, & 0 < \theta < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = \sum_{i=1}^{n} X_i$, find the posterior p.d.f , $f(\theta|Y)$.

(iv) (3%) Take the loss function to be $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$; find the Bayes' solution $\delta(y)$ for a point estimate of θ.

(v) (5%) Find $E(Y)$ and $Var(Y)$.