For the following problems , $P_n$ denotes the vector space of all polynomials with real coefficients of degree less than or equal to $n$, $V^\perp$ denotes the orthogonal compliment of vector space $V$, and $A^T$ denotes the transpose of the matrix $A$.

1. Let $V = \{[x_1, x_2] | x_1$ is a real number and $x_2$ is a positive number $\}$ with addition defined by $[x_1, x_2] \oplus [y_1, y_2] = [x_1 + y_1 + 1, x_2 y_2]$ and with scalar multiplication defined by $r[x_1, x_2] = [rx_1 + r - 1, (x_2)^r]$. ( for example $[1,3] \oplus [-3,7] = [-1,21]$ ,$2[1,3] = [2+2-1, 3^2] = [3,9]$ )
   (a) Find the additive identity $\vec{0}$ and the additive inverse of $\vec{v} = [3,2]$. (b) Show the scalar multiplication satisfies the distributive property $r(\vec{x} \oplus \vec{y}) = r\vec{x} \oplus r\vec{y}$. (13%)

2. Find a basis for the subspace $V = \{p(x) | p(x) = x^4 p(\frac{-1}{x})$ for $p(x) \in P_4\}$. (7%)

3. Let $V = \{p(x) | p(x)$ is a polynomial without constant term $\}$ be the subspace of polynomial space $P$ with inner product $< p(x), q(x) >= \int_0^1 x p(x) q(x)\, dx$. Show that $V^\perp = \{0\}$. (5%)

4. The matrix $A$ is row-equivalent to the matrix $B$.

$$A = \begin{bmatrix} 1 & a_1 & 1 & 0 & b_1 \\ -1 & a_2 & 1 & 0 & b_2 \\ 3 & a_3 & -1 & 0 & b_3 \\ 1 & a_4 & 0 & 1 & b_4 \\ 2 & a_5 & 3 & -1 & b_5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) Find a basis for the nullspace of $A$. (5%)
   (b) Find vectors $\vec{a} = [a_1, a_2, \ldots, a_5]^T$ and $\vec{b} = [b_1, b_2, \ldots, b_5]^T$. (5%)
   (c) Find bases for the row space and the column space of $A$ , respectively. (5%)

5. Let $\vec{v}$ be a column vector in $R^n$. Define the matrix $A = I - \alpha \vec{v}\vec{v}^T$ . Find the value of $\alpha$ so that $A^{-1} = A$. Solve the linear system $A\vec{x} = \vec{b}$ where $\vec{v} = [1,0,2,0,-1]^T$ and $\vec{b} = [0,11,-1,9,1]^T$. (10%)

6. Let $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ be an orthogonal basis for $P_n$ with a given inner product denoted by $< \cdot, \cdot >$. If each $\phi_k(x)$ is a monic ( the leading coefficient is 1) polynomial of degree $k$, show that $\phi_{k+1}(x) - x\phi_k(x) = -\alpha_k \phi_k(x) - \beta_k \phi_{k-1}(x)$, where $\alpha_k = \frac{<x\phi_k(x), \phi_k(x)>}{<\phi_k(x), \phi_k(x)>}$ and $\beta_k = \frac{<\phi_k(x), \phi_k(x)>}{<\phi_{k-1}(x), \phi_{k-1}(x)>}$ for $k = 1, 2, \ldots, n-1$. (8%)

7. Let $\{1, x, x^2 - \frac{1}{3}, x^3 - \frac{3}{5}x, \phi_4(x)\}$ be an orthogonal basis for $P_4$ with respect to the inner product $< p(x), q(x) >= \int_{-1}^{1} p(x) q(x)\, dx$. Find $\phi_4(x)$. (7%)

8. Consider the vector space $P_2$ of polynomials of degree at most 2, and let $T : P_2 \to P_2$ be the linear transformation such that $T(x^2 - 1) = -x^2 + 1$, $T(x) = x^2 - x - 1$ and $T(1) = x^2 - 3x + 1$.
   (a) Find a matrix representation for $T$ associated with the ordered basis $\{x^2 - 1, x, 1\}$. (5%)
   (b) Find $p(x)$ such that $T(p(x)) = x + 2$. (5%)
   (c) Find eigenvalues $\lambda$ and the associated eigenfunctions $p(x)$ for $T$. (i.e. $T(p(x)) = \lambda p(x)$ ) (5%)
   (d) Find $T^8(x^2 - 2x + 1)$ (5%)

9. Determine whether the statement is true or false. If it is true, prove it, otherwise, give a counter example. (15%)
   (a) If $S$ is a subspace of an inner product space V, then $(S^\perp)^\perp = S$.
   (b) Let $T : R^n \to R^n$ be a linear transformation. Let the set $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ be linearly independent in $R^n$, then the set $\{T(\vec{x}_1), T(\vec{x}_2), \ldots, T(\vec{x}_k)\}$ is also linearly independent.
   (c) Let $\vec{v}$ be a unit column vector in $R^n$. The characteristic polynomial $(\det(xI - A))$ of the matrix $A = \vec{v}\vec{v}^T$ is $x^{n-1}(x-1)$.