There are six problems and 100 points in total.

1. (12 pts.) Let W be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{a}_1 = [1, 1, -1]$, $\mathbf{a}_2 = [0, 1, -2]$, $\mathbf{a}_3 = [2, 3, -4]$, and $\mathbf{a}_4 = [0, 3, -6]$.
   (a) Determine whether the vector $\mathbf{b} = [1, -1, 3]$ lies in W.
   (b) Find a basis for W.

2. (18 pts.) Let $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & 0 & -3 & 1 \\ 4 & 1 & -3 & 0 \end{bmatrix}$.
   (a) Determine the rank of A.
   (b) Find a basis for the column space and a basis for the row space of A.
   (c) Find a basis for the nullspace of A.

3. (18 pts.) Let A and B be $n \times n$ matrices.
   (a) Prove that $\text{rank}(AB) \le \text{rank}(A)$.
   (b) Prove that $\text{rank}(AB) \le \text{rank}(B)$.
   (c) Find a necessary and sufficient condition for $\text{rank}(AB) = \text{rank}(B)$.

4. (12 pts.) Let T be the linear operator on the vector space $P_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ given by

$$T(f(x)) = f(1) + f'(0)x + 2f''(0)x^2.$$

   (a) Give the matrix of T relative to the standard basis $\{1, x, x^2\}$ of P_2.
   (b) Is T diagonalizable?

5. (20 pts.) Suppose that a 4×4 matrix A has eigenvalues 1, -1, 2, and -2. Find the trace and determinant of $A^6 - 5A^4 + 4I$.

6. (20 pts.) Find a unitary matrix U and a diagonal matrix D such that $U^{-1}AU = D$, where $A = \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.