(20%) 1. Let f be a real-valued function defined on $[a, b]$ and $x_0 \in [a, b]$. Show that the following two properties are equivalent.

(a) for any $\varepsilon > 0$, there exists a constant $\delta > 0$ so that for $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$, we have $|f(x) - f(x_0)| < \varepsilon$.

(b) for any sequence $\{c_n\}_{n=1}^{\infty} \subset [a, b]$ satisfying $\lim_{n\to\infty} c_n = x_0$, we have $\lim_{n\to\infty} f(c_n) = f(x_0)$

(20%) 2. Let $\{a_n\}_{n=1}^{\infty}$ be a real-valued sequence satisfying

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 0.$$

Set $f(x) = \sum_{n=1}^{\infty} a_n x^n$, for $x \in (-\infty, \infty)$. Show that

(a) f is continuous in $(-\infty, \infty)$, and

(b) the derivative f' is also continuous in $(-\infty, \infty)$ and we have

$$f'(x) = \sum_{n=1}^{n} n a_n x^{n-1}, \quad \text{for } x \in (-\infty, \infty).$$

(20%) 3. Let $\{a_n\}_{n=1}^{\infty}$ be a real-valued sequence and $\sum_{n=1}^{\infty} |a_n|$ exist. Set $f(x) = \sum_{n=1}^{\infty} a_n \cos nx$, for $x \in [-\pi, \pi]$. Show that

(a) the function f is continuous on $[-\pi, \pi]$, and

(b) the integral

$$\int_{-\pi}^{\pi} f^2(x)dx = \pi \sum_{n=1}^{\infty} a_n^2.$$

(20%) 4. Show the following

(a) for $\varepsilon \in (0, 1]$, we have $\lim_{\varepsilon\to 0} \varepsilon^{\frac{1}{100}} \ln \varepsilon = 0$,

(b) $\lim_{x\to\infty} \dfrac{x^{100}}{e^x} = 0$,

and find the value of

(c) $\int_0^1 x^{-\frac{99}{100}} \ln x dx = ?$,

(d) $\int_0^{\infty} e^{-x} \cos(5x) dx = ?$.

(20%) 5. Let f be a bounded variation function defined on $[a, b]$. Show that

(a) there exist two increasing functions g and h defined on $[a, b]$ so that

$$f = g - h ,$$

and

(b) f is continuous on $[a, b]$ except possibly countable many points.

There are six problems and 100 points in total.

1. (12 pts.) Let W be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{a}_1 = [1, 1, -1]$, $\mathbf{a}_2 = [0, 1, -2]$, $\mathbf{a}_3 = [2, 3, -4]$, and $\mathbf{a}_4 = [0, 3, -6]$.

   (a) Determine whether the vector $\mathbf{b} = [1, -1, 3]$ lies in W.

   (b) Find a basis for W.

2. (18 pts.) Let $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & 0 & -3 & 1 \\ 4 & 1 & -3 & 0 \end{bmatrix}$.

   (a) Determine the rank of A.

   (b) Find a basis for the column space and a basis for the row space of A.

   (c) Find a basis for the nullspace of A.

3. (18 pts.) Let A and B be $n \times n$ matrices.

   (a) Prove that $\text{rank}(AB) \leq \text{rank}(A)$.

   (b) Prove that $\text{rank}(AB) \leq \text{rank}(B)$.

   (c) Find a necessary and sufficient condition for $\text{rank}(AB) = \text{rank}(B)$.

4. (12 pts.) Let T be the linear operator on the vector space $P_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ given by

$$T(f(x)) = f(1) + f'(0)x + 2f''(0)x^2.$$

   (a) Give the matrix of T relative to the standard basis $\{1, x, x^2\}$ of P_2.

   (b) Is T diagonalizable?

5. (20 pts.) Suppose that a 4×4 matrix A has eigenvalues 1, -1, 2, and -2. Find the trace and determinant of $A^6 - 5A^4 + 4I$.

6. (20 pts.) Find a unitary matrix U and a diagonal matrix D such that $U^{-1}AU = D$, where $A = \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Show all your work.

1. (10 pts.) A light is 4 miles from a straight shoreline. The light revolve at the rate of 2 rev/min. Find the speed of the spot of light along the shore when the light spot is 2 miles past the point on the shore closest to the source of light.

2. (10 pts.) Sketch the graph of $f(x)=e^{-x}\sin x$ for $x\geq 0$, and determine as many as possible of the key features such as range, intercepts, relative extrema, inflection points, asymptotes, and concavity.

3. (10 pts.) Minimize $f(x,y,z)=2x^2+3y^2+4z^2$ subject to $x+y=4$ and $x-2y+5z=3$.

4. (10 pts.) Find the equation for the tangent line to the curve

$$y=F(x)=\int_1^{\sqrt{x}}\frac{t^2+t+1}{\sqrt{3t^2+1}}dt \quad \text{at } x=1.$$

5. (10 pts.) Find the volume of the solid D bounded below by the paraboloid $z=x^2+y^2$ and above by the plane $2x+z=3$.

6. (10 pts.) Let $f(x)=\begin{cases} x^2\sin\frac{1}{x} & \text{for } x\neq 0 \\ 0 & \text{for } x=0 \end{cases}$.

   (1) Use the definition of the derivative to prove that f is differentiable at $x=0$.

   (2) Prove or disprove that $f'(x)$ is continuous at $x=0$.

7. (10 pts.) Let $a_n=\left(1+\frac{1}{n}\right)^n$, $n=1,2,3,\ldots$.

   (1) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ converges.

   (2) Approximate $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ correct to four decimal places.

8. (10 pts.) Let $f(x,y,z)=z(x-y)^5+xy^2z^3$.

   (1) Find the directional derivative of f at (2, 1, -1) in the direction of the outward normal to the sphere $x^2+y^2+z^2=6$.

   (2) In what direction is the directional derivative at (2, 1, -1) largest?

9. (20 pts.) Define $F(s)=L\{f(t)\}=\int_0^{\infty}e^{-st}f(t)\,dt$. And, the inverse $L^{-1}\{F(s)\}$ is the function $f(t)$ such that $L\{f(t)\}=F(s)$.

   (1) Show that $L\{e^{at}f(t)\}=F(s-a)$ and $L\{tf(t)\}=-F'(s)$.

   (2) Find $L\{t^n\}$, $L\{\cos at\}$ (with $s-a>0$), $L\{t\cos 2t\}$, and $L^{-1}\left\{\frac{5}{s}\right\}$.

For the following problems , P_n denotes the vector space of all polynomials with real coefficients of degree less than or equal to n, $V^\perp$ denotes the orthogonal compliment of vector space V, and A^T denotes the transpose of the matrix A.

1. Let $V = \{[x_1, x_2] | x_1$ is a real number and x_2 is a positive number $\}$ with addition defined by $[x_1, x_2] \oplus [y_1, y_2] = [x_1 + y_1 + 1, x_2 y_2]$ and with scalar multiplication defined by $r[x_1, x_2] = [rx_1 + r - 1, (x_2)^r]$. (for example $[1,3] \oplus [-3,7] = [-1,21]$,$2[1,3] = [2+2-1, 3^2] = [3,9]$)
(a) Find the additive identity $\vec{0}$ and the additive inverse of $\vec{v} = [3,2]$. (b) Show the scalar multiplication satisfies the distributive property $r(\vec{x} \oplus \vec{y}) = r\vec{x} \oplus r\vec{y}$. (13%)

2. Find a basis for the subspace $V = \{p(x) | p(x) = x^4 p(\frac{-1}{x})$ for $p(x) \in P_4\}$. (7%)

3. Let $V = \{p(x) | p(x)$ is a polynomial without constant term $\}$ be the subspace of polynomial space P with inner product $< p(x), q(x) >= \int_0^1 x p(x) q(x)\, dx$. Show that $V^\perp = \{0\}$. (5%)

4. The matrix A is row-equivalent to the matrix B.

$$A = \begin{bmatrix} 1 & a_1 & 1 & 0 & b_1 \\ -1 & a_2 & 1 & 0 & b_2 \\ 3 & a_3 & -1 & 0 & b_3 \\ 1 & a_4 & 0 & 1 & b_4 \\ 2 & a_5 & 3 & -1 & b_5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) Find a basis for the nullspace of A. (5%)
   (b) Find vectors $\vec{a} = [a_1, a_2, \ldots, a_5]^T$ and $\vec{b} = [b_1, b_2, \ldots, b_5]^T$. (5%)
   (c) Find bases for the row space and the column space of A , respectively. (5%)

5. Let $\vec{v}$ be a column vector in R^n. Define the matrix $A = I - \alpha \vec{v}\vec{v}^T$. Find the value of α so that $A^{-1} = A$. Solve the linear system $A\vec{x} = \vec{b}$ where $\vec{v} = [1,0,2,0,-1]^T$ and $\vec{b} = [0,11,-1,9,1]^T$. (10%)

6. Let $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ be an orthogonal basis for P_n with a given inner product denoted by $< \cdot, \cdot >$. If each $\phi_k(x)$ is a monic (the leading coefficient is 1) polynomial of degree k, show that $\phi_{k+1}(x) - x\phi_k(x) = -\alpha_k \phi_k(x) - \beta_k \phi_{k-1}(x)$, where $\alpha_k = \frac{<x\phi_k(x), \phi_k(x)>}{<\phi_k(x), \phi_k(x)>}$ and $\beta_k = \frac{<\phi_k(x), \phi_k(x)>}{<\phi_{k-1}(x), \phi_{k-1}(x)>}$ for $k = 1, 2, \ldots, n-1$. (8%)

7. Let $\{1, x, x^2 - \frac{1}{3}, x^3 - \frac{3}{5}x, \phi_4(x)\}$ be an orthogonal basis for P_4 with respect to the inner product $< p(x), q(x) >= \int_{-1}^{1} p(x) q(x)\, dx$. Find $\phi_4(x)$. (7%)

8. Consider the vector space P_2 of polynomials of degree at most 2, and let $T : P_2 \to P_2$ be the linear transformation such that $T(x^2 - 1) = -x^2 + 1$, $T(x) = x^2 - x - 1$ and $T(1) = x^2 - 3x + 1$.

   (a) Find a matrix representation for T associated with the ordered basis $\{x^2 - 1, x, 1\}$. (5%)
   (b) Find $p(x)$ such that $T(p(x)) = x + 2$. (5%)
   (c) Find eigenvalues λ and the associated eigenfunctions $p(x)$ for T. (i.e. $T(p(x)) = \lambda p(x)$) (5%)
   (d) Find $T^8(x^2 - 2x + 1)$ (5%)

9. Determine whether the statement is true or false. If it is true, prove it, otherwise, give a counter example. (15%)

   (a) If S is a subspace of an inner product space V, then $(S^\perp)^\perp = S$.
   (b) Let $T : R^n \to R^n$ be a linear transformation. Let the set $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ be linearly independent in R^n, then the set $\{T(\vec{x}_1), T(\vec{x}_2), \ldots, T(\vec{x}_k)\}$ is also linearly independent.
   (c) Let $\vec{v}$ be a unit column vector in R^n. The characteristic polynomial ($\det(xI - A)$) of the matrix $A = \vec{v}\vec{v}^T$ is $x^{n-1}(x-1)$.

(10%) 1. Let $f:(0,+\infty)\to R$ be differentiable and $\lim\limits_{x\to+\infty} f'(x)=5$.

Find $\lim\limits_{x\to+\infty}(f(x+5)-f(x))$.

(20%) 2. Evaluate the following limit:

(5%) (a) $\lim\limits_{x\to 0}\dfrac{e^x-1-x-\dfrac{x^2}{2}}{x^3}$.

(5%) (b) $\lim\limits_{x\to 1^-}\dfrac{\dfrac{\pi}{2}-\sin^{-1}(x)}{x-1}$.

(5%) (c) $\lim\limits_{x\to 0^+} x^{\frac{1}{2+\ln(x)}}$.

(5%) (d) $\lim\limits_{x\to -\infty} x^2(1-x\sin(\frac{1}{x}))$.

(20%) 3. Evaluate the following integral:

(5%) (a) $\int_2^4 \frac{1+x}{1-x}dx$.

(5%) (b) $\int_{\pi/4}^{\pi/2}\frac{\cos^3(x)}{\sqrt{\sin(x)}}dx$.

(5%) (c) $\int_1^e \ln^2(x)dx$.

(5%) (d) $\int_0^2 \frac{1}{1+e^x}dx$.

(10%) 4. Let $\Gamma(\alpha)=\int_0^{+\infty} e^{-x}x^{\alpha-1}dx$, $\alpha\in(0,+\infty)$. Prove

(5%) (a) $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, $\forall\alpha>0$.

(5%) (b) $\Gamma(n)=(n-1)!$, $\forall n\in N$.

(20%) 5. Let $A=\begin{bmatrix}3 & 2 & 1\\ 0 & 1 & 2\\ 0 & 1 & -1\end{bmatrix}$.

(5%) (a) Find A^{-1}.

(15%) (b) Find the eigenvalues of A and a non-singular matrix, P, such that $P^{-1}AP=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}$.

(10%) 6. Show that if $A^2=A$, and if λ is an eigenvalue of A, then either $\lambda=1$ or $\lambda=0$.

(10%) 7. Let $A=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$. Find a symmetric matrix, B, such that $x^T Ax = x^T Bx$, $\forall x \in R^2$.

1. (35%) Let $X_1,\ldots,X_n$ be iid with pdf

$$f(x|\theta)=\begin{cases}\theta x^{\theta-1} & \text{if } 0\le x\le 1, 0<\theta<\infty,\\ 0 & \text{elsewhere}\end{cases}$$

(i) (7%) Find the maximum likelihood estimator (MLE) of θ. Is the MLE of θ unbiased? Show the details.

(ii) (4%) Find the Rao-Cramer's lower bound of MLE of θ.

(iii) (3%) Let W be Beta(1,1), and consider $Y=-\theta\log W$, where $\theta>0$. Find the cdf of Y.

(iv) (6%) Suppose the random sample $Y_1,\ldots,Y_n$ from the distribution of Y. Find the MLE and the minimum variance unbiased estimator (MVUE) of $P(Y\le 2)$).

(v) (5%) Suppose the random sample $Y_1,\ldots,Y_n$ from the distribution of Y. Find a uniformly most powerful critical region of size $\alpha=0.05$ and $n=2$ for testing $H_0:\theta=2$ against $H_a:\theta>2$. Show the details. ($\chi^2_{(2)}=5.991,\chi^2_{(3)}=7.815,\chi^2_{(4)}=9.488,\chi^2_{(5)}=11.071,\chi^2_{(6)}=12.592$)

(vi) (4%) Let the random sample $W_1,\ldots,W_n$ from Beta(1,1) and let $W_{(1)},\ldots,W_{(n)}$ denote the order statistics. Find the distribution of the j^{th} order statistic $W_{(j)}$. Show the details.

(vii) (6%) Let the random sample $W_1,\ldots,W_n$ from Beta(1,1) and let $W_{(1)},\ldots,W_{(n)}$ denote the order statistics. The range is defined as $R=W_{(n)}-W_{(1)}$ and the mid-range is defined as $V=(W_{(1)}+W_{(n)})/2$. Find the distribution of R.

2. (8%) Consider X and Y have a trinomial distribution with joint pmf

$$p(x,y) = \frac{n!}{x!y!(n-x-y)!} p_1^x p_2^y p_3^{n-x-y},$$

where x and y are nonnegative integers with $x + y \leq n$;
$p_1,\ p_2,\ p_3 \in (0,1)$ and $p_1 + p_2 + p_3 = 1$; and let $p(x,y) = 0$ elsewhere.

   (i) (4%) Find the moment generating function of a trinomial distribution.

   (ii) (4%) Compute $E(Y|X = x)$.

3. (13%) Let $X \sim N(0,\ \theta)$ where $0 < \theta < \infty$.

   (i) (3%) Show that the family $N(0,\ \theta)$ where $0 < \theta < \infty$ is not complete by finding at least one nonzero function $u(x)$ such that E(u(X))=0, for all $\theta > 0$.

   (ii) (6%) If $X_1,\ X_2, \cdots,\ X_n$ be a random sample from $N(0,\ \theta)$, show that the MLE of θ is an efficient estimate of θ.

   (iii) (4%) Find the minimum variance unbiased estimator of θ^2.

4. (20%) For the simple linear regression model, $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, where $i = 1, \ldots, n$ and ϵ_i are i.i.d $N(0, \sigma^2)$.

   (i) (4%) To derive the least squared estimator(s) (LSE) for β_0 and β_1.

   (ii) (5%) Find the sampling distribution of $\hat{\beta}_1$. Show the details.

   (iii) (5%) For testing $H_0 : \beta_1 = 0$. Find the distribution of $SSReg/\sigma^2$ under H_0 is true, where $SSReg$ is the sum of squares regression.

   (iv) (6%) Derive the likelihood ratio test for testing $H_0 : \beta_1 = 0$ against $H_a : \beta_1 \neq 0$.

5. (24%) Let $X_1, \ldots, X_n$ be a random sample from a distribution, where $Pr(X_i = 1) = \theta$, $Pr(X_i = 0) = 1 - \theta$, where $0 < \theta < 1$.

   (i) (5%) Show that this is a uniformly most powerful test when we test $H_0 : \theta = \frac{1}{2}$ against $H_a : \theta < \frac{1}{2}$.

   (ii) (5%) Based on (i). Use the Central Limit Theorem (CLT) to find n and c so that the significance level is approximately 0.10 and the power of the test is approximately 0.80 when $\theta = \frac{1}{3}$.

   (iii) (6%) If the prior p.d.f of Θ is

$$f(\theta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, & 0 < \theta < 1 \\ 0, & \text{otherwise.} \end{cases}$$

   Let $Y = \sum_{i=1}^{n} X_i$, find the posterior p.d.f , $f(\theta|Y)$.

   (iv) (3%) Take the loss function to be $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$; find the Bayes' solution $\delta(y)$ for a point estimate of θ.

   (v) (5%) Find $E(Y)$ and $Var(Y)$.