國立政治大學 100 學年度研究所博碩士班招生考試試題

考試科目	線性代數	所別	應用數學	考試時間	2 月 26 日(六)第 2 節

Please show all your work.

1. **(10%)** Let V be a finite-dimensional vector space over a field F, and let $T: V \to V$ be linear. If rank (T) = rank (T^2), prove that $R(T) \cap N(T) = \{0\}$. ($R(T)$ is the range of T and $N(T)$ is the kernel of T.)

2. Compute the specified expression.
   (a) **(10%)** Express A as its LU factorization where

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}.$$

   (b) **(5%)** Express A in a diagonal form $S\Lambda S^{-1}$ where Λ is a diagonal matrix with eigenvalues of A on its diagonal and

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

   (c) **(5%)** Express A as its QR factorization where $A = \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$.

3. Let $C[0,2\pi]$ denote the inner product space of real continuous functions defined on $[0,2\pi]$. For any $f, g \in [0,2\pi]$, the inner product is defined as

$$< f,g >= \int_0^{2\pi} f(x)g(x)dx.$$

   (a) **(10%)** Let W be a subspace of $C[0,2\pi]$ where $W = \text{span}\{1, \ \sin(x), \ \sin^2(x/2)\}$. Provide an orthonormal basis of W.
   (b) **(5%)** Let $h(x) \in W$ such that $h(x)$ minimizes $\|h(x) - x\|$ where $\|\bullet\|$ is the norm induced by the inner product. Express $h(x)$ as the linear combination of the basis obtained in (a).

4. Let A be a $m \times n$ matrix with rank(A) = n and $m > n$.
   (a) **(10%)** Show that $A^T A$ is a nonsingular matrix.
   (b) **(10%)** Show that $\text{rank}(AA^T) \le \text{rank}(A)$ and deduce that AA^T is singular.

備註	試題隨卷繳交	請注意：背面還有試題。

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5. A square matrix A is called **nilpotent** if there exists a positive integer k such that $A^k = 0$.
   (a) **(5%)** What are the possible eigenvalues of a nilpotent matrix? Justify your answer.
   (b) **(10%)** Show that a nonzero nilpotent matrix is not diagonalizable.

6. **(20%)** Let A be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For any given complex vector x, define $\rho(x) = \dfrac{x^H A x}{x^H x}$. Show that $\max_{x \neq 0} \rho(x) = \lambda_1$ and $\min_{x \neq 0} \rho(x) = \lambda_n$.

備註	試題隨卷繳交