Please show all your work.

- 1. (10%) Let V be a finite-dimensional vector space over a field F, and let $T:V\to V$ be linear. If rank $(T) = \operatorname{rank}(T^2)$, prove that $R(T) \cap N(T) = \{0\}$. (R(T) is the range of T and N(T) is the kernel of T.)
- 2. Compute the specified expression.
 - (a) (10%) Express A as its LU factorization where

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}$$

(b) (5%) Express A in a diagonal form $S\Lambda S^{-1}$ where Λ is a diagonal matrix with eigenvalues of A on its diagonal and

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

- (c) (5%) Express A as its QR factorization where $A = \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$
- 3. Let $C[0,2\pi]$ denote the inner product space of real continuous functions defined on $[0,2\pi]$. For any $f, g \in [0, 2\pi]$, the inner product is defined as

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx$$

- (a) (10%) Let W be a subspace of $C[0,2\pi]$ where $W = \text{span}\{1, \sin(x), \sin^2(x/2)\}$. Provide an orthonormal basis of W.
- (b) (5%) Let $h(x) \in W$ such that h(x) minimizes ||h(x) x|| where $||\cdot||$ is the norm induced by the inner product. Express h(x) as the linear combination of the basis obtained in (a).
- 4. Let A be a $m \times n$ matrix with rank(A) = n and m > n
 - (a) (10%) Show that $A^T A$ is a nonsingular matrix.
 - (b) (10%) Show that $rank(AA^T) \le rank(A)$ and deduce that AA^T is singular.

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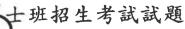
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國立政治大學 100 學年度研究所領土班招生考試試題 第2頁,共2頁考試科目線性代數 所別應用數學 考試時間 2月26日(分第2節



- 5. A square matrix A is called **nilpotent** if there exists a positive integer k such that $A^k = 0$.
 - (a) (5%) What are the possible eigenvalues of a nilpotent matrix? Justify your answer.
 - (b) (10%) Show that a nonzero nilpotent matrix is not diagonalizable.
- 6. (20%) Let A be a Hermitian matrix with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. For any given complex vector x, define $\rho(x) = \frac{x^H A x}{x^H x}$. Show that $\max_{x \neq 0} \rho(x) = \lambda_1$ and $\min_{x \neq 0} \rho(x) = \lambda_n$.

