國立中央大學104學年度碩士班考試入學試題

所別：統計研究所碩士班 不分組(一般生)　科目：基礎數學　共 2 頁 第 1 頁
統計研究所碩士班 不分組(在職生)

本科考試可使用計算器，廠牌、功能不拘　　*請在答案卷（卡）內作答

參考用

1. (24%) Let f and g be real-valued functions. Prove that

$$\int |f(x)g(x)|dx \le \left(\int |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int |f(x)|^q dx\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$. .

(a) (8%) Specifically, $p=2$, $q=2$, show *Cauchy-Schwarz inequality*.

$$\left(\int f(x)g(x)dx\right)^2 \le \left(\int f^2(x)dx\right)\left(\int g^2(x)dx\right).$$

(b) (8%) *Young's inequality*: Let $a>0$ and $b>0$ and $\frac{1}{p}+\frac{1}{q}=1$, $1<p,q<\infty$. Prove that

$$ab \le \frac{a^p}{p}+\frac{b^q}{q}.$$

(Hint: Use the convex function $f(x)=e^x$ and choose $a^p=e^x$ and $b^q=e^y$.)

(c) (8%) Use *Young's inequality*. Prove that

$$\int |f(x)g(x)|dx \le \left(\int |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int |f(x)|^q dx\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

2. (16%) Find the minimum distance from a point on the following surfaces to the origin:

(a) (8%) $x+y=z$,

(b) (8%) $xy+2xz=5\sqrt{5}$.

3. (14%) For all $x>0$, let $f(x)=\int_x^\infty e^{-\frac{u^2}{2}}du$ and $g(x)=\frac{x}{1+x^2}e^{-\frac{x^2}{2}}$. Prove that

(a) (7%)

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=1,$$

(b) (7%) $g(x)<f(x)$.

4. (16%) If α, $\beta>0$ and

$$G=\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

(a) (8%) Show $G=B\Lambda B^{-1}$.

(b) (8%) Evaluate $e^G=\sum_{n=0}^\infty \frac{1}{n!}G^n$.

5. (15%) A matrix of the form $H=I-2\frac{vv^T}{v^Tv}$ is called a Householder transform, where I is identity and v is a non-zero column vector. v^T is the transpose of v.

(a) (5%) Evaluate H^2.

(b) (5%) Show that $Hv=-v$.

(c) (5%) Show that if $x^Tv=0$, then $Hx=x$.

注意：背面有試題

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6. (15%) Given a full rank matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix},$$

find the eigenvalues for H_i and show trace H_i = rank H_i, $i = 1, \cdots, 3$,

(a) (5%) $H_1 = A(A^T A)^{-1} A^T$,

(b) (5%) $H_2 = H_1^2$,

(c) (5%) $H_3 = (I - H_1)$,

where A^T is the transpose of A and A^{-1} is the inverse of A.

參考用

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