考試科目	線性代數 81112、81162	所別	應用數學系 8111	考試時間	3月1日(日)第二節

1. (20%) Let a, b be two distinct real numbers. Let

$$A = \begin{bmatrix} a^3 & a^2 & a & 1 \\ 3a^2 & 2a & 1 & 0 \\ b^3 & b^2 & b & 1 \\ 3b^2 & 2b & 1 & 0 \end{bmatrix}.$$

Determine all possible values of $\operatorname{rank}(A)$.

2. (20%) Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 1 & 7 \\ 1 & 0 & 3 \end{bmatrix}.$$

(a) Determine if A is diagonalizable or not.

(b) Find matrices Q and D such that D is either a diagonal matrix or a Jordan block matrix, satisfying

$$D = Q^{-1}AQ.$$

3. (20%) Let V, W be two vector spaces over a field $\mathbb{F}$, and let $T: V \to W$ be a linear transformation of T. Suppose U is a subspace of W. Is $T^{-1}(U)$ a subspace of V? Justify your answer.

4. (20%) Let $P_3(\mathbb{R})$ be the set of all real polynomials of degrees at most 3. We define $\langle \cdot, \cdot \rangle : P_3(\mathbb{R}) \times P_3(\mathbb{R}) \to \mathbb{R}$ by

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\,dx,$$

for all $f(x), g(x) \in P_3(\mathbb{R})$.

(a) What are the conditions need to be satisfied for $\langle \cdot, \cdot \rangle$ to be an inner product on $P_3(\mathbb{R})$?

(b) Suppose that $\langle \cdot, \cdot \rangle$ is indeed a inner product on $P_3(\mathbb{R})$. Apply the Gram-Schmid process to $\{1, x, x^2, x^3\}$ to find an orthogonal basis for $P_3(\mathbb{R})$.

5. (20%) Let V be a finite-dimensional vector space, and let T be a linear operator on V. Suppose that $\operatorname{rank}(T) = \operatorname{rank}(T^2)$.

(a) Show that $R(T) = R(T^2)$ and $N(T) = N(T^2)$. ($R(T)$ is the range of T and $N(T)$ is the null space of T.)

(b) Show that $V = R(T) \oplus N(T)$.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。