國立臺灣師範大學104學年度碩士班招生考試試題

科目：機率與統計　　　　　　　　　　　　　　　　適用系所： 數學系

注意: 1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則不予計分。

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1. (20分) John and Mary both visit a convenient store once every day to collect miniatures. John wants to have "Snoopy" in series A, whereas Mary is interested in "Hello Kitty" in series B. Miniatures in series A and B are randomly distributed to the customers independently. Suppose every day "Snoopy" can be obtained with probability p, and "Hello Kitty" can be obtained with probability q. Let X denote the number of days it takes for John to get his first "Snoopy" and Y the number of days it takes for Mary to get her first "Hello Kitty".

   (a) Find the probability distribution of X.

   (b) Find the probability distribution of the minimum of X and Y, i.e., $\min(X, Y)$.

   (c) Find the probability of $X = Y$, i.e., $\mathrm{P}(X = Y)$.

   (d) Find the expectation of X given $X \leq Y$, i.e., $\mathrm{E}(X|X \leq Y)$.

2. (20分) Let Y be a uniform random variable on $(0, 1)$. For a given $Y = y$, X follows a binomial distribution, $\text{Binomial}(10, y)$.

   (a) Find the probability distribution of X.

   (b) Given $x = 8$, find the conditional probability density function of Y.

3. (20分) If gene frequencies are in equilibrium, the three genotypes AA, Aa, and aa occur with probabilities θ^2, $2\theta(1-\theta)$ and $(1-\theta)^2$, $0 < \theta < 1$, respectively. A random sample of n people are taken, and the numbers observed for the three genotypes are, respectively, X, Y and Z with $X + Y + Z = n$,

   (a) Find the maximum likelihood estimator of θ.

   (b) Consider the test statistic $T = 2X + Y$. Given that T has a binomial distribution with parameters $2n$ and θ. Find a uniformly most powerful (UMP) test, with the level of significance α, for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ for some $\theta_1 < \theta_0$ based on the test statistic T.

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4. (20分) Let $X_1, X_2, \cdots, X_n$ be random variable from a population with probability density function $f(x;\theta) = e^{(\theta - x)}$, where $x > \theta$, $-\infty < \theta < \infty$, and $f(x;\theta) = 0$, otherwise. Let Y denote the smallest order statistic.

   (a) Show that Y is a sufficient statistic for θ.

   (b) If $Y - c$ is used to estimate θ for some constant c. Find the value of c that minimises the mean square error of $Y - c$.

5. (20分) Consider a simple linear regression for $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, $i = 1, 2, \cdots .n$, where x_i's are fixed constants and $\varepsilon_i \sim^{\text{iid}}$ Normal$(0, \sigma^2)$. The least square estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are used to estimate the intercept and slope, respectively. Let $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$ be the fitted values.

   (a) Show that $\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} \hat{Y}_i$.

   (b) Let SSR be the regression sum of squares $\text{SSR} = \sum_{i=1}^{n} (\hat{Y}_i - \overline{Y})^2$, where $\overline{Y}$ is the mean of Y. Find the expectation of SSR/1.

   (c) Which of the following statement is true. 「可單選或複選，全對才給分」

   ⓐ Residuals $e_i = Y_i - \hat{Y}_i$, are statistically independent with Y_i.

   ⓑ Residuals are statistically independent with $\hat{Y}_i$.

   ⓒ $\hat{\beta}_0$ and $\hat{\beta}_1$ are statistically indepdendent.

   ⓓ $\hat{\beta}_1$ and $\overline{Y}$ are statistically indepdendent.

   ⓔ The fitted line always passes through the point $(\overline{x}, \overline{Y})$.

   (d) If a 95% confidence interval for β_1 was (-5.65, 2.61). Which of the following statement is true. 「可單選或複選，全對才給分」

   ⓐ Reject the null hypothesis at $\alpha = 0.05$ for testing $H_0 : \beta_1 = 0$ versus $H_1 : \beta_1 \neq 0$.

   ⓑ Do not reject the null hypothesis at $\alpha = 0.05$ for testing $H_0 : \beta_1 = 0$ versus $H_1 : \beta_1 \neq 0$.

   ⓒ When constructing the confidence interval for β_1, the degrees of freedom for the t-value is $n - 1$.

   ⓓ When constructing the confidence interval for β_1, the degrees of freedom for the t-value is $n - 2$.

   ⓔ We could conclude that there is no relationship between Y and x at $\alpha = 0.05$.

(試題結束)