國立臺灣師範大學 104 學年度碩士班招生考試試題

科目：基礎數學　　　　適用系所：數學系

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則不予計分。

Part I : Calculus

1. (7 points) Find the equation of the tangent line to $y = (\ln x)^{\cos x}$ at $(e, 1)$.

2. (7 points) In using the ϵ, δ definition to prove that $\lim_{x\to 0} x^2 = 1$, where $\epsilon = 1$, what is the largest value that δ can have?

3. (7 points) Suppose $f(x) = (x^x)^x$. Find $f'(x)$.

4. (7 points) Calculate $\lim_{\theta\to 0} \dfrac{\sin(\cos\theta)}{\sec\theta}$.

5. (7 points) Calculate $\displaystyle\int_1^3 \frac{dx}{\sqrt{x}(1+x)}$.

6. (7 points) Suppose $g(x) = \sin(x^3)$. Find $g^{(9)}(0)$.

7. (8 points) Determine whether the vector filed is conservative. If it is, find a potential function.

   (a) $\mathbf{F}(x, y) = e^x(\cos y\,\mathbf{i} - \sin y\,\mathbf{j})$

   (b) $\mathbf{F}(x, y) = \dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{x^2+y^2}}$

   (c) $\mathbf{F}(x, y, z) = xy^2z^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k}$

   (d) $\mathbf{F}(x, y, z) = ye^z\mathbf{i} + ze^x\mathbf{j} + xe^y\mathbf{k}$

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Part II : Linear algebra

8. Let $A = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & -1 & -3 \end{bmatrix}$

   (a) (5 points) Find the LU decomposition of A.

   (b) (7 points) Let $b_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \end{bmatrix}$, $b_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 5 \end{bmatrix}$, $b_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 6 \end{bmatrix}$, $b_4 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 7 \end{bmatrix}$.
   Decide for which b_i such that $Ax = b_i$ is consistent, then solve the linear system.

9. (8 points) Let V be the plane in $\mathbb{R}^3$ defined by the equation $x-2y+z=0$. Find the projection matrix on V.

10. Let P_3 denote the vector space of polynomials of degree at most 3. Consider the linear transformation $T : P_3 \to P_3$ given by

$$T(f)(t) = 2f(t) + (1-t)f'(t).$$

   (a) (4 points) Give the matrix representation of T with respect to the ordered basis $\{1, t-1, (t-1)^2, (t-1)^3\}$.

   (b) (6 points) Determine $\operatorname{Ker}(T)$ and $\operatorname{image}(T)$. Give your reasoning.

11. (10 points) If $a_0 = 0,\ a_1 = a_2 = 1$, and $a_{k+1} = 2a_k + a_{k-1} - 2a_{k-2}$, for $k \geq 2$, determine the formula for a_k.

12. Let x, y be vectors in $\mathbb{R}^n$.

   (a) (6 points) Prove the Cauchy-Schwarz Inequality by minimizing the quadratic function $Q(t) = \|x - ty\|^2$, for $t \in \mathbb{R}$.

   (b) (4 points) Prove the triangle inequality : $\|x+y\| \leq \|x\| + \|y\|$.

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