# 國立臺灣師範大學 104 學年度碩士班招生考試試題

科目：**線性代數與代數** 適用系所：**數學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則不予計分。

Part I : Linear algebra

1. Consider the ordered bases $\epsilon = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$,

$\alpha = \left\{ v_1 = \begin{bmatrix}1\\1\\0\\2\end{bmatrix}, v_2 = \begin{bmatrix}1\\2\\1\\2\end{bmatrix}, v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, v_4 = \begin{bmatrix}2\\0\\1\\5\end{bmatrix} \right\}$ and

$\beta = \left\{ w_1 = \begin{bmatrix}1\\1\\0\\2\end{bmatrix}, w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, w_3 = \begin{bmatrix}4\\6\\3\\7\end{bmatrix}, w_4 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \right\}$ for $\mathbb{R}^4$. Let $T$ be a linear transformation on $\mathbb{R}^4$ with $T(v_i) = w_i$, $i = 1, 2, 3, 4$.

   (a) (4 points) Let $A = \begin{bmatrix}1&1&1&2\\1&2&1&0\\0&1&1&1\\2&2&1&5\end{bmatrix}$, find the inverse of $A$.

   (b) (9 points) Find the matrix of $T$ with respect to $\alpha, \epsilon$, the matrix of $T$ with respect to $\alpha, \beta$ and the standard matrix of $T$.

   (c) (7 points) Find the coordinate of $\begin{bmatrix}3\\2\\3\\4\end{bmatrix}$ with respect to $\alpha$ then find $T\left(\begin{bmatrix}3\\2\\3\\4\end{bmatrix}\right)$.

   (d) (10 points) Find $\ker(T)$, $\ker(T)^{\perp}$ and $\operatorname{rank}(T)$. Give your reasoning.

2. (10 points) Let $A = \begin{bmatrix}2&2&-2\\2&-1&-1\\-2&-1&-1\end{bmatrix}$. Find an orthonormal basis consisting of eigenvectors of $A$.

3. (10 points) Let $A$ be an $n \times n$ real matrix. Prove that if $\lambda$ is an eigenvalue of $A$ with geometric multiplicity $d$, then $\lambda$ is also an eigenvalue of $A^T$ with geometric multiplicity $d$, where $A^T$ is the transpose of $A$.

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Part II : Algebra

4. (12 points) Let $G$ be the symmetric group $S_5$.

   (a) Find the number of elements of order 5 in $G$.

   (b) Find the number of Sylow 5-subgroups in $G$.

5. (10 points) Let $G$ be a group and let $M, H$ be subgroups of $G$. Suppose that $N$ is a normal subgroup of $M$. If $M/N$ is abelian, prove that $(M \cap H)/(N \cap H)$ is abelian.

6. (10 points) Let $\mathbb{Q}$ be the field of rational numbers and $\mathbb{Q}[x]$ be the ring of polynomials over $\mathbb{Q}$. Determine if the polynomial $x^8+4x+1$ is irreducible in $\mathbb{Q}[x]$.

7. (18 points) Let $\mathbb{Z}$ be the ring of integers. Prove or disprove the following statements.

   (a) 2 is an irreducible element in $\mathbb{Z}[\sqrt{-7}]$.

   (b) 2 is a prime in $\mathbb{Z}[\sqrt{-7}]$.

   (c) $\mathbb{Z}[\sqrt{-7}]$ is a principal ideal domain.

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