國立臺灣師範大學 104 學年度碩士班招生考試試題

科目：**高等微積分**　　　　適用系所：**數學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則不予計分。

1. (a) (2 points) Find the Maclaurin series for $\sin x$.

   (b) (8 points) Approximate $\sin(0.1)$ to within 10^{-3}. (求 $\sin(0.1)$ 的近似值，使得誤差小於 10^{-3}。)

2. (10 points) Find the integral $\int e^{\sqrt{x}}\,dx$.

3. (10 points) Consider the function $f(x,y) = e^{x\sin y}$. Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\lim_{(h,k)\to(0,0)} \frac{\left|f(2+h, \frac{\pi}{3}+k) - e - T(h,k)\right|}{\sqrt{h^2+k^2}} = 0.$$

4. (10 points) Let $B_2 = \{(x,y) \mid x^2+y^2 \le 4\}$ be the closed ball of radius 2, centered at the origin in $\mathbb{R}^2$. Evaluate the double integral

$$\iint_{B_2} \frac{1}{x^2+y^2+1}\,dA.$$

5. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function that satisfies

   (i) $f(0) = 1/2$;

   (ii) $0 \le f'(x) \le 1/3$, for all $x \in [0,1]$.

   (a) (5 points) Show that $f(1) \in [1/2, 5/6]$.

   (b) (10 points) Show that there exists a unique $x^* \in (0,1)$ such that

$$f(x^*) = x^*.$$

   (c) (2 points) Use the information from the definition of f and problems (a) (b) to sketch a graph of $y = f(x)$ and $y = x$, for $0 \le x \le 1$.

   (d) (10 points) Let the sequence $\{x_n\}_{n=0}^{\infty}$ be defined by

$$x_{n+1} = f(x_n), \quad x_0 = 0.$$

   Show that the sequence $\{x_n\}_{n=0}^{\infty}$ is increasing and is bounded above by x^*.

   (e) (3 points) Find $\lim_{n\to\infty} x_n$.

(背面尚有試題)

6. Consider the fuction $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined as follows:

$$(u, v, w) = F(x, y, z) = (x + y + z,\ xy + yz + zx,\ xyz).$$

It is clear that $F(1, 2, 3) = (6, 11, 6)$.

(a) (9 points) Show that in some neighborhood of $(1, 2, 3)$, F has a differentiable inverse G.

(b) (6 points) Continued from (a), write $(x, y, z) = G(u, v, w)$. Compute $\frac{\partial x}{\partial w}(6, 11, 6)$.

7. (15 points) Let $B = \{(x, y) \mid x^2 + y^2 \leq 1\}$ be the closed unit ball in $\mathbb{R}^2$. Suppose that there is a collection $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of open balls whose union covers B; that is,

$$B \subset \bigcup_{\alpha \in A} U_\alpha.$$

Prove that the same collection $\mathcal{U}$ also covers an open ball which is slightly larger than B; that is, there is a positive number $\delta > 0$ such that

$$\{(x, y) \mid x^2 + y^2 < (1 + \delta)^2\} \subset \bigcup_{\alpha \in A} U_\alpha.$$

(試題結束)