國立臺灣大學 104 學年度碩士班招生考試試題

(1) (20%) Let α be a regular curve in $\mathbb{R}^3$ parametrized by its arc length. Suppose that its curvature and torsion are both nonzero everywhere. Prove that

$$\frac{\tau}{\kappa} + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)' = 0 \qquad \text{(Prime means taking derivative.)}$$

iff α is spherical, i.e. it lies entirely on a sphere.

(2) (25%) Consider the punctured disk $D^* = \{(x,y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1\}$ with the metric

$$\frac{8}{(x^2+y^2)\big(\log(x^2+y^2)\big)^2}(\mathrm{d}x^2 + \mathrm{d}y^2)\ .$$

Prove that

$$\mathrm{Area}(\square) = 4\pi - 2(i_1 + i_2 + i_3 + i_4)$$

for a geodesic quadrilateral $\square$ in D^*, where i_1, i_2, i_3 and i_4 are the interior angles at the vertices of $\square$. (A quadrilateral is a polygon with four edges and four vertices.)

(3) Let S be a surface in $\mathbb{R}^3$, and p be a point on S. Let $\{\mathrm{v}, \mathrm{w}\}$ be an orthonormal basis for $\mathrm{T}_p(S)$. Consider

$$\gamma(s) = \exp_p(s\,\mathrm{v})\ ,$$
$$\psi(s,t) = \exp_{\gamma(s)}(t\,\mathrm{w}(s))$$

where $\mathrm{w}(s)$ is the parallel transport of w along $\gamma(s)$.

(a) (5%) Prove that there exists some $\varepsilon > 0$ such that $\psi : \{(s,t) \mid s^2 + t^2 < \varepsilon^2\} \to S$ is a diffeomorphism onto an open neighborhood of p in S.

(b) (15%) Let $E(s,t)$, $F(s,t)$ and $G(s,t)$ be the coefficients of the first fundamental form in terms of the coordinate system given by ψ. Namely, $E(s,t) = \langle \frac{\partial\psi}{\partial s}, \frac{\partial\psi}{\partial s}\rangle$, $F(s,t) = \langle \frac{\partial\psi}{\partial s}, \frac{\partial\psi}{\partial t}\rangle$ and $G(s,t) = \langle \frac{\partial\psi}{\partial t}, \frac{\partial\psi}{\partial t}\rangle$. Show that

$$G(s,t) \equiv 1\ ,\ E(s,0) = 1\ ,\ F(s,0) = 0\ ,\ \frac{\partial E}{\partial t}(s,0) = 0\ ,\ \text{and}\ \frac{\partial F}{\partial t}(s,0) = 0\ .$$

(c) (10%) Prove that $\dfrac{\partial^2\sqrt{E}}{\partial t^2}(s,0) = -\mathrm{K}_{\gamma(s)}$, the Gaussian curvature of S at $\gamma(s)$.

(4) (a) (15%) Let S be a closed surface in $\mathbb{R}^3$. Prove that there exists a point $p \in S$ such that the Gaussian curvature of S at p is positive, $\mathrm{K}_p > 0$.

(b) (10%) Does there exist closed minimal surfaces in $\mathbb{R}^3$? Explain your answer.

試題隨卷繳回