

科目：數學(以離散數學、線性代數為主)

編號：413

適用：資工系

考生注意：

1. 依次序作答，只要標明題號，不必抄題。

2. 答案必須寫在答案卷上，否則不予計分。

3. 限用藍、黑色筆作答；試題須隨卷繳回。

本試題

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(以下各題均須寫出計算或證明過程方予計分)

1. (10%) How many bytes contains

(a) (5%) exactly two 1's;

(b) (5%) at least six 1's?

2. (10%) Establish the following for all $n \geq 1$ by the Principle of MathematicalInduction: $\sum_{i=1}^n i(2^i) = 2 + (n-1)2^{n+1}$ 3. (10%) Let S be a set of five positive integers the maximum of which is at most 9. Prove that the sums of the elements in all the nonempty subsets of S cannot all be distinct.4. (10%) For $n \geq 0$, let a_n count the number of ways a sequence of 1's and 2's will sum to n . For example, $a_3 = 3$ because (1) 1, 1, 1; (2) 1, 2; and (3) 2, 1 sum to 3. Find and solve a recurrence relation for a_n .5. (10%) Let $m, n \in \mathbf{Z}^+$ with $m \geq n \geq 2$.(a) (5%) Determine how many distinct cycles of length 4 there are in $K_{m,n}$.(b) (5%) How many different paths of length 2 are there in $K_{m,n}$?6. (15%) Let $T_C: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be a linear operator with multiplication matrix $C = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.(a) (4%) T_C maps the line $y = 2x + 1$ into another line. Find its equation.(b) (2%) Find the rank (the dimension of the range of T_C) and the nullity (the dimension of the kernel of T_C).(c) (6%) We know that C is the standard matrix for the linear operator T_C relative to the standard basis $B_1 = \{(1, 0), (0, 1)\}$ for \mathbf{R}^2 . Suppose that $B_2 = \{(1, 1), (1, 2)\}$ is the new basis for \mathbf{R}^2 . Find the multiplication matrix D , such that T_C and T_D denote the same linear operator and the two matrices C and D are called similar.(d) (3%) Let $v \in \mathbf{R}^2$ and denote the coordinate vectors of v relative to the bases B_1 and B_2 respectively as $[v]_{B_1}$ and $[v]_{B_2}$. Find the transition matrix P , such that $[v]_{B_2} = P[v]_{B_1}$.

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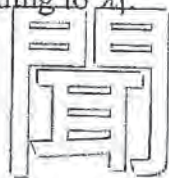
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7. (25%) Let $A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix}$.

- (a) (3%) Find the eigenvalues λ_1 and λ_2 , $\lambda_1 \leq \lambda_2$ of A .
- (b) (5%) Find an orthonormal basis for the eigenspace corresponding to λ_1 .
- (c) (3%) Find an orthonormal basis for the eigenspace corresponding to λ_2 .
- (d) (3%) Find a matrix $P = [u_1 \ u_2 \ u_3]$ that orthogonally diagonalizes A , where u_1 , u_2 , and u_3 are the unit eigenvectors of A in column form.
- (e) (6%) From (d), show that $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T$, which is called a *spectral decomposition* of A .
- (f) (5%) Let $v = (1, 2, 3) \in \mathbb{R}^3$. Find the orthogonal projection of v on the eigenspace corresponding to λ_1 .



8. (10%) Let $F(-\infty, \infty)$ be the vector space of real-valued functions. Suppose that W is a subspace of $F(-\infty, \infty)$ and W is spanned by $\varphi_1(t)$ and $\varphi_2(t)$, where

$$\varphi_1(t) = \sqrt{2} \cos(2\pi t) \text{ and } \varphi_2(t) = \sqrt{2} \sin(2\pi t), \quad 0 \leq t \leq 1. \text{ Let } W \text{ have the inner}$$

product $\langle p(t), q(t) \rangle \triangleq \int_0^1 p(t)q(t)dt$, where $p(t), q(t) \in W$; and the *norm*(or *length*) of $p(t)$

is defined by $\|p(t)\| = \sqrt{\langle p(t), p(t) \rangle}$.

- (a) (5%) Show that $\|\varphi_1(t)\| = \|\varphi_2(t)\| = 1$ and $\langle \varphi_1(t), \varphi_2(t) \rangle = 0$. Hint:

$$\sin^2 \theta = (1 - \cos 2\theta) / 2, \quad \cos^2 \theta = (1 + \cos 2\theta) / 2, \quad \text{and} \quad \sin 2\theta = 2 \sin \theta \cos \theta.$$

- (b) (5%) Suppose $s_1(t) = \frac{\sqrt{2}}{2} \varphi_1(t) + \frac{\sqrt{2}}{2} \varphi_2(t)$ and $s_2(t) = \frac{-\sqrt{2}}{2} \varphi_1(t) + \frac{\sqrt{2}}{2} \varphi_2(t)$. Show that $\|s_1(t) - s_2(t)\|^2 = 2$.

