科目：數學(以離散數學、線性代數為主)

編號：413　適用：資工系

考生注意：
1.依次序作答，只要標明題號，不必抄題。
2.答案必須寫在答案卷上，否則不予計分。
3.限用藍、黑色筆作答；試題須隨卷繳回。

(以下各題均須寫出計算或證明過程方予計分)

1. (10%) How many bytes contains
   (a) (5%) exactly two 1's;
   (b) (5%) at least six 1's?

2. (10%) Establish the following for all $n \geq 1$ by the Principle of Mathematical Induction: $\sum_{i=1}^{n} i(2^i) = 2 + (n-1)2^{n+1}$.

3. (10%) Let S be a set of five positive integers the maximum of which is at most 9. Prove that the sums of the elements in all the nonempty subsets of S cannot all be distinct.

4. (10%) For $n \geq 0$, let a_n count the number of ways a sequence of 1's and 2's will sum to n. For example, $a_3 = 3$ because (1) 1, 1, 1; (2) 1, 2; and (3) 2, 1 sum to 3. Find and solve a recurrence relation for a_n.

5. (10%) Let $m, n \in \mathbf{Z}^+$ with $m \geq n \geq 2$.
   (a) (5%) Determine how many distinct cycles of length 4 there are in $K_{m,n}$.
   (b) (5%) How many different paths of length 2 are there in $K_{m,n}$?

6. (15%) Let $T_C : R^2 \to R^2$ be a linear operator with *multiplication matrix* $C = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.
   (a) (4%) T_C maps the line $y = 2x+1$ into another line. Find its equation.
   (b) (2%) Find the *rank* (the dimension of the range of T_C) and the *nullity* (the dimension of the kernel of T_C).
   (c) (6%) We know that C is the standard matrix for the linear operator T_C relative to the standard basis $B_1 = \{(1, 0), (0, 1)\}$ for R^2. Suppose that $B_2 = \{(1, 1), (1, 2)\}$ is the new basis for R^2. Find the multiplication matrix D, such that T_C and T_D denote the same linear operator and the two matrices C and D are called *similar*.
   (d) (3%) Let $v \in R^2$ and denote the coordinate vectors of v relative to the bases B_1 and B_2 respectively as $[v]_{B_1}$ and $[v]_{B_2}$. Find the *transition matrix* P, such that $[v]_{B_2} = P[v]_{B_1}$.

7. (25%) Let $A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix}$.

(a) (3%) Find the eigenvalues (λ_1 and λ_2, $\lambda_1 \le \lambda_2$) of A.

(b) (5%) Find an orthonormal basis for the eigenspace corresponding to λ_1.

(c) (3%) Find an orthonormal basis for the eigenspace corresponding to λ_2.

(d) (3%) Find a matrix $P = [\mathrm{u}_1\ \mathrm{u}_2\ \mathrm{u}_3]$ that orthogonally diagonalizes A, where u_1, u_2, and u_3 are the unit eigenvectors of A in column form.

(e) (6%) From (d), show that $A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_1 \mathbf{u}_2 \mathbf{u}_2^T + \lambda_2 \mathbf{u}_3 \mathbf{u}_3^T$, which is called a *spectral decomposition* of A.

(f) (5%) Let $v = (1, 2, 3) \in R^3$. Find the orthogonal projection of v on the eigenspace corresponding to λ_1.

8. (10%) Let $F(-\infty,\infty)$ be the vector space of real-valued functions. Suppose that W is a subspace of $F(-\infty,\infty)$ and W is spanned by $\varphi_1(t)$ and $\varphi_2(t)$, where $\varphi_1(t) = \sqrt{2}\cos(2\pi t)$ and $\varphi_2(t) = \sqrt{2}\sin(2\pi t)$, $0 \le t \le 1$. Let W have the inner product $\langle p(t), q(t)\rangle \triangleq \int_0^1 p(t)q(t)dt$, where $p(t), q(t) \in W$; and the *norm*(or *length*) of $p(t)$ is defined by $\|p(t)\| = \sqrt{\langle p(t), p(t)\rangle}$.

(a) (5%) Show that $\|\varphi_1(t)\| = \|\varphi_2(t)\| = 1$ and $\langle\varphi_1(t), \varphi_2(t)\rangle = 0$. Hint:

$\sin^2\theta = (1-\cos 2\theta)/2$, $\cos^2\theta = (1+\cos 2\theta)/2$, and $\sin 2\theta = 2\sin\theta\cos\theta$.

(b) (5%) Suppose $s_1(t) = \frac{\sqrt{2}}{2}\varphi_1(t) + \frac{\sqrt{2}}{2}\varphi_2(t)$ and $s_2(t) = \frac{-\sqrt{2}}{2}\varphi_1(t) + \frac{\sqrt{2}}{2}\varphi_2(t)$. Show that $\|s_1(t) - s_2(t)\|^2 = 2$.