(以下各題均須寫出計算或證明過程方予計分)

1. For any given set in a tennis tournament, opponent A can beat opponent B in seven ways. (At 6-6 they play a tie breaker.) The first opponent to win three sets wins the tournament.
   (a)(5%) In how many ways can scores be recorded with A winning in five sets?
   (b) (5%) In how many ways can scores be recorded with the tournament requiring at least four sets?

2. (10%) Let $A = \{a_1, a_2, a_3, a_4, a_5\} \subseteq \mathbf{Z}^+$. Prove that A contains a nonempty subset S where the sum of the elements in S is a multiple of 5. (Here it is possible to have a sum consisting of only one summand.)

3. (10%) Let $A \subseteq \{1, 2, 3, \ldots, 50\}$ where $|A| = 10$. For any subset B of A let s_B denote the sum of the elements in B. Prove that there are distinct subsets C, D of A such that $|C| = |D| = 4$ and $s_C = s_D$.

4. (10%) For $n \geq 1$, let a_n be the number of ways to write n as an ordered sum of positive integers, where each summand is at least 2. (For example, $a_5 = 3$ because here we may represent 5 by 5, by 2 + 3, by 3 + 2.) Find and solve a recurrence relation for a_n.

5. Let $G = (V, E)$ be a loop-free undirected graph. Define the relation $\mathcal{R}$ on E as follows: If $e_1, e_2 \in E$, then $e_1 \,\mathcal{R}\, e_2$ if $e_1 = e_2$ or if e_1 and e_2 are edges of a cycle C in G.
   (a)(7%) Verify that $\mathcal{R}$ is an equivalence relation on E.
   (b) (3%) Describe the partition of E induces by $\mathcal{R}$.

6. (10%) Prove that if A is a singular $n \times n$ matrix, then the reduced row-echelon form of A has at least one row of zeros.

7. (10%) Let $A=\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & 1 & \frac{1}{5} \\ \frac{1}{6} & \frac{1}{7} & 1 \end{bmatrix}$. Describe what happens to the matrix A^k when k is allowed to increase indefinitely (that is, as $k \to \infty$).

8. (20%)

 (a) (10%) Let the vector space P_2 have the inner product $\langle p,q \rangle = \int_0^1 p(x)q(x)dx$. Apply the *Gram-Schmidt process* to transform the standard basis $S = \{1, x, x^2\}$ into an orthonormal basis. (P_2: the set of all real polynomials of degree 2 or less)

 (b) (5%) For a polynomial $2x+1 \in P_2$, find the coordinate vector of the polynomial relative to the orthonormal basis obtained in (a).

 (c) (5%) Find the length of polynomial $2x+1$ directly from the result of (b).

9. (10%) Let $A=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ be an orthogonal matrix. Suppose $\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = A\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$, where $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ and $\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$ are the coordinate vectors relative to the old orthonormal basis $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ and a new basis, respectively. Find the new basis.