注意事項：

1. 本試題共 10 題，配分共 100 分。
2. 請標明大題、子題編號作答，不必抄題。
3. 全部答案均須在答案卷之答案欄內作答，否則不予計分。

1. (10%) Use mathematical induction to prove that

(a) (5%) $\sum_{i=1}^{n} i(i+1)(i+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

(b) (5%) $L_n = F_{n-1} + F_{n+1}$, where L_k is the kth Lucas number ($L_0=2$, $L_1=1$, and $L_k=L_{k-1}+L_{k-2}$, $k \geq 2$) and F_k is the kth Fibonacci number ($F_0=0$, $F_1=1$, and $F_k=F_{k-1}+F_{k-2}$, $k \geq 2$).
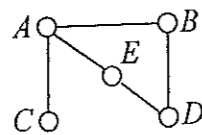
2. (10%) Solve the following recurrence relation.
$$a_n = 6a_{n-1} - 9a_{n-2}, \ n \geq 2, \ a_0=2, \ a_1=3.$$

3. (10%) Solve the following recurrence relation by giving big-O notation bound.
$$T(n)=2T(n/2)+n \ , \ n \geq 2, \ T(1)=1.$$

4. (10%) Consider the following graph.

(a) (5%) Determine if the graph has an Euler Circuit. If it does, write it down. If it doesn't, explain why.

(b) (5%) Determine if the graph has a Hamiltonian Path. If it does, write it down. If it doesn't, explain why.

5. (10%) For each of the following algebraic systems, determine with proof whether or not it is a semigroup. Suppose that $\mathbb{Z}$ is the set of all integer numbers.

(a) (5%) A pair $<\mathbb{Z}, \oplus>$ with $a \oplus b = a+b+1$ for $a, b \in \mathbb{Z}$.

(b) (5%) A pair $<\mathbb{Z}, \otimes>$ with $a \otimes b = a+b+ab$ for $a, b \in \mathbb{Z}$.

6. (10%) Determine whether or not the following binary relations are partial orders. Explain your answer.
(a) (5%) {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)} on the set A = {1, 2, 3}.
(b) (5%) {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 3)} on the set A = {1, 2, 3}.

7. (10%) Consider each of the following functions. Is it one-to-one? Is it onto? Explain your answer. Suppose that $\mathbb{Z}^*$ is the set {0, 1, 2, 3, ...} and $\mathbb{R}$ is the set of all real numbers.
(a) (5%) $f: \mathbb{Z}^* \times \mathbb{Z}^* \rightarrow \mathbb{Z}^*$, $f(x, y) = x^2 + 2y$.
(b) (5%) $g: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $g(x, y) = x^2 + y^2$.

8. (10%) How many integer solutions are there to each of the following equations?
(a) (5%) $x_1 + x_2 + x_3 + x_4 = 20$ and $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.
(b) (5%) $x_1 + x_2 + x_3 + x_4 = 20$ and $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$.

9. (10%) Suppose a connected planar graph G has vertices of degrees 3, 3, 4, 5, and 5.
(a) (5%) Find the number of edges of G.
(b) (5%) Find the number of faces (regions) of G.

10. (10%) Consider the poset <{2, 4, 5, 10, 12, 20, 25}, |>, where the relation | is such that $x|y$ if x divides y.
(a) (5%) Draw the Hasse diagram for this poset.
(b) (5%) How many maximal chains does the Hasse diagram of part (a) have? (A maximal chain is one that is not a subset of another chain.)