Notation.

I_n : the identity matrix of size n.

$M_{n \times m}(\mathbb{R})$: the set of $n \times m$ real matrices.

1 (10) Let $A, B \in M_{n \times n}(\mathbb{R})$. Prove that rank$(AB) \leq$ rank(A).

2 Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & -1 & -5 & 1 & -6 \end{bmatrix}.$$

　a. (10) Find $M \in M_{5 \times 5}(\mathbb{R})$ with rank$(M) = 2$ such that $AM = 0$.

　b. (10) Suppose that $B \in M_{5 \times 5}(\mathbb{R})$ such that $AB = 0$. Prove that rank$(B) \leqslant 2$.

3 Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}.$$

　a. (10) Find the eigenvalues and eigenvectors of A.

　b. (5) Find the eigenvalues and eigenvectors of $A^3 + A^2 + A + I_3$.

4 (10) Let $f(t)$ be the characteristic polynomial of A. Suppose that A is diagonalizable then show that $f(A) = 0$.

5 (10) Prove that similar matrices have the same characteristic polynomial.

6 Let $W = \text{span}\{[1, 0, 1], [2, 1, 1]\}$.

　a. (5) Find an orthonormal basis for W.

　b. (10) Find the matrix that projects vectors in $\mathbb{R}^3$ on W.

7 (10) Find the least-squares fit to the data points, $(1, 2)$, $(2, 3)$, $(3, 5)$, $(4, 7)$, by a linear function $f(x) = r_0 + r_1 x$.

8 (10) Find the general solution of the differential equation

$$\begin{aligned} x' &= 8x + 10y \\ y' &= -5x - 7y. \end{aligned}$$