國立中山大學104學年度碩士暨碩士專班招生考試試題

科目名稱：工程數學甲【電機系碩士班甲組、丁組、己組】　　　　題號：431002

※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘）（問答申論題）　　

Problem 1 (20%)

Let u be a solution to the heat equation: $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

with boundary conditions: $u(0,x) = f(x),\ 0 \le x \le 1$, and $\dfrac{\partial u}{\partial x}(t,0) = \dfrac{\partial u}{\partial x}(t,1) = 0,\ 0 \le t < \infty$.

(a) (10%) Define the thermal energy $\mathcal{T}(t) = \int_0^1 u(t,x)dx$. Show that under the above assumptions, $\mathcal{T}(t)$ is constant in time; i.e., $\mathcal{T}(t) = \mathcal{T}(0) = \int_0^1 f(x)dx$, for all $t \ge 0$.

(b) (10%) Let $f(x) = \cos(\pi x)$. Find the solution u.

Problem 2 (25%)

Let $\mathbf{F} = (y^2 + axz + yz)\mathbf{i} + (z^2 + bxy + xz)\mathbf{j} + (x^2 + cyz + xy)\mathbf{k}$.

(a). (10%) Find the values of the constants $a,\ b,\ c$ for which $\mathbf{F}$ is conservative.

(b). (15%) For the values found in (a), find a surface S with the following property: the path integral $\int_P^Q \mathbf{F} \cdot d\mathbf{r}$ is equal to 0 for any two points $P,\ Q$ (connected by any curve C) on the surface S.

Problem 3 (13%)

下面的問題共有(a)~(c)三個子題；**每個子題都只要寫出提問的答案即可**(不須寫出答案背後的推導)。

Let $\alpha \ne 0$ and $A \in \mathbb{R}^{n\times n}$, and let $\Lambda(A)$ denote the set of all eigenvalues of A.

(a) Suppose $I + \alpha A$ is nonsingular, thus, for any nonzero scalar α, $\Omega_\alpha := (I - \alpha^{-1}A)(I + \alpha A)^{-1}$ is a well-defined $\mathbb{R}^{n\times n}$ matrix. Then we know from knowledge of eigensystem of a square matrix that, corresponding to any $\lambda \in \Lambda(A)$, there is a $\mu \in \Lambda(\Omega_\alpha)$. What is the mathematical relation between λ and μ? (3%)

(b) If Ω_1, that is $(I - A)(I + A)^{-1}$, is an orthogonal matrix, then what mathematical relation between A and A^T can be derived? (5%)

(c) If Ω_α is an idempotent matrix, then what are all possible values of $\det A$? (5%)

Problem 4 (12%)

本問題共有(a)、(b)兩個子題，**每個子題都只要寫出提問的答案即可**(不須寫出答案背後的推導)。

(a) Write out the set

$S := \left\{ P = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, [\alpha\ \beta\ \gamma]^T \in N([1\ -2\ -1])\ \ but\ \ [\alpha\ \beta\ \gamma]^T \notin \left(R\left([1\ 0\ -1]^T\right)\right)^\perp \right\}$, where $N(\bullet)$ and $R(\bullet)$ indicate the null space and the range of a matrix, respectively. (5%)

(b) Consider the inner product space $V = \left(\mathbb{R}^{2\times 2}, \langle \bullet, \bullet \rangle\right)$, where $\langle A, B \rangle := tr(A^T B)$ for A and B in $\mathbb{R}^{2\times 2}$. Describe $S^\perp$ as the span of a set of orthonormal vectors in $\mathbb{R}^{2\times 2}$. (7%)

Problem 5 (15%)

Let C be a circle $|z|=2$ described in the counterclockwise direction.

(a)(5%) Compute the following integral

$$\oint_C \frac{e^{kz^n}}{z} dz, \quad n \text{ is a positive integer}$$

(b)(10%) Suppose the answer you obtained in Part (a) is $jn\pi$. Use Part (a) to evaluate

$$\int_0^{2\pi} e^{2k\cos(n\theta)} \sin\left(2k\sin(n\theta)\right) d\theta .$$

Problem 6 (15%)

Define the Fourier transform of a signal $f(t)$ as $F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$, and its inverse Fourier transform is $f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$. It is already known that the Fourier transform of signal $x(t)=\sin(at)/(\pi t)$ is

$$X(j\omega)=\begin{cases}1, & |\omega|<a \\ 0, & |\omega|>a\end{cases}$$

and $\mathscr{F}\{tf(t)\}=j\frac{d}{d\omega}F(j\omega)$. Compute the Fourier transform of the signal

$$x(t)=t\left(\frac{\sin(t)\sin(t/2)}{\pi t^2}\right).$$