# 國立中山大學104學年度碩士暨碩士專班招生考試試題

科目名稱：工程數學乙【電機系碩士班乙組】 題號：431001

※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘）（問答申論題） 共2頁第1頁

**Problem 1** (25%)

Consider the differential equation: $\dot{x}(t) = ax(t) + u(t)$, with initial condition $x(0) = x_0$.

(a) (10%) Show that the solution of the equation is $x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}u(\tau)d\tau,\ t \geq 0$.

(b) (10%) Suppose $a < 0$. Show that for any bounded $u$ and any initial condition $x_0$, the corresponding solution $x$ is bounded.

(c) (5%) Suppose $a > 0$ and $u(t) = \sin(t)$. Find the initial condition $x_0$ such that the corresponding solution $x$ is bounded. In this case, what is $x$?

**Problem 2** (20%)

本問題係由兩個小題所組成，此兩小題共含有(a)~(f)六個子題；**每個子題都只要寫出提問的答案即可**(不須寫出答案背後的推導)。

- (5%) Consider the set $S := \{ A \in \mathbb{R}^{2\times2} \mid A = A^T \ and \ tr(A) = k \}$.

(a) What are all possible values of $k$ so that $S$ is a subspace of $V := \{ A \in \mathbb{R}^{2\times2} \mid A = A^T \}$? (2%)

(b) Consider the inner product space $(S, \langle\bullet,\bullet\rangle)$ with $\langle A, B\rangle := tr(A^T B)$ for $A$ and $B$ in $S$. Find an orthogonal basis for $S$. (3%)

- (15%) Let $\alpha \neq 0$ and $A \in \mathbb{R}^{n\times n}$, and let $\Lambda(A)$ denote the set of all eigenvalues of $A$.

(c) What is the condition on $\Lambda(A)$ so that $I + \alpha A$ is nonsingular? (2%)

(d) Suppose $I + \alpha A$ is nonsingular, thus, for any nonzero scalar $\alpha$, $\Omega_\alpha := (I - \alpha^{-1}A)(I + \alpha A)^{-1}$ is a well-defined $\mathbb{R}^{n\times n}$ matrix. Then we know from knowledge of eigensystem of a square matrix that, corresponding to any $\lambda \in \Lambda(A)$, there is a $\mu \in \Lambda(\Omega_\alpha)$. What is the mathematical relation between $\lambda$ and $\mu$? (3%)

(e) If $\Omega_1$, that is $(I - A)(I + A)^{-1}$, is an orthogonal matrix, then what mathematical relation between $A$ and $A^T$ can be derived? (5%)

(f) If $\Omega_\alpha$ is an idempotent matrix, then what are all possible values of $\det A$? (5%)

**Problem 3** (25%)

本問題共有(a)~(d)四個子題，除了(b)子題裡的 **discuss** 之外，**每個子題都只要寫出提問的答案即可**(不須寫出答案背後的推導)。

(a) Describe the set of all $\mathbb{R}^{3\times1}$ vectors $[\alpha\ \beta\ \gamma]^T$ that satisfy the two conditions

$[\alpha\ \beta\ \gamma]^T \in N([1\ -2\ -1])$ and $[\alpha\ \beta\ \gamma]^T \notin \left(R([1\ 0\ -1]^T)\right)^\perp$, where $N(\bullet)$ and $R(\bullet)$ indicate the null space and the range of a matrix, respectively. (5%)

(b) Now denote the set

$$S := \left\{ P = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \in \mathbb{R}^{2\times2} \;\middle|\; [\alpha\ \beta\ \gamma]^T \in N([1\ -2\ -1]) \ but \ [\alpha\ \beta\ \gamma]^T \notin \left(R([1\ 0\ -1]^T)\right)^\perp \right\}$$ in terms of solution of (a). Write out the set $S$ *and* **discuss** if the closure property of vector addition holds for set $S$, i.e. whether the implication "$P_1, P_2 \in S \Rightarrow P_1 + P_2 \in S$" holds for any $P_1$ and $P_2$. (1+5%)

(c) Consider the inner product space $V = (\mathbb{R}^{2\times2}, \langle\bullet,\bullet\rangle)$, where $\langle A, B\rangle := tr(A^T B)$ for $A$ and $B$ in $\mathbb{R}^{2\times2}$. Describe $S^\perp$ as the span of a set of orthonormal vectors in $\mathbb{R}^{2\times2}$. (7%)

(d) Now let $T := Span\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$ be a subspace of $V$ and let $P$ be any vector of $S$. What is the distance of $P$ to $T$? (7%)

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**Problem 4** (15%)

Let $C$ be a circle $|z|=2$ described in the counterclockwise direction.

(a)(5%) Compute the following integral

$$\oint_C \frac{e^{kz^n}}{z} dz, \quad n \text{ is a positive integer}$$

(b)(10%) Suppose the answer you obtained in Part (a) is $jn\pi$. Use Part (a) to evaluate

$$\int_0^{2\pi} e^{2k\cos(n\theta)} \sin\left(2k\sin(n\theta)\right) d\theta.$$

**Problem 5** (15%)

Define the Fourier transform of a signal $f(t)$ as $F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$, and its inverse Fourier transform is $f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$. It is already known that the Fourier transform of signal $x(t)=\sin(at)/(\pi t)$ is

$$X(j\omega)=\begin{cases}1, & |\omega|<a \\ 0, & |\omega|>a\end{cases}$$

and $\mathscr{F}\{tf(t)\}=j\frac{d}{d\omega}F(j\omega)$. Compute the Fourier transform of the signal

$$x(t)=t\left(\frac{\sin(t)\sin(t/2)}{\pi t^2}\right).$$