ANSWER *any 5* QUESTIONS FROM BELOW, EACH OF WHICH CARRIES 20 POINTS.

1. (a) (10 points) Give its standard matrix representation of the linear transformation T if T is defined by

$$T([x_1, x_2, x_3]) = x_1 + 2x_2 + 3x_3.$$

   (b) (10 points) Find the general matrix representation for the reflection of the plane in the line $y = 2x$.

2. (20 points) Let $T : R^3 \to R^3$ defined by $T([x, y, z]) = [x+y, x+z, y-z]$. Let $B = ([1,1,1], [1,1,0], [1,0,0])$ and $E = ([1,0,0], [0,1,0], [0,0,1])$ be two ordered bases of $\mathbb{R}^3$. Find the matrix representations $R_B = [T]_B$ and $R_E = [T]_E$ of T with respect to bases B and E, respectively. Find also an invertible matrix C such that $R_E = C^{-1}R_BC$.

3. Let

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}.$$

   (a) (5 points) Find the characteristic polynomial.

   (b) (5 points) Find the real eigenvalues and the corresponding eigenvectors.

   (c) (10 points) Find an matrix C and a diagonal matrix D such that $D = C^{-1}AC$.

4. (20 points) Let $y_0, y_1, y_2, \ldots$ be the sequence of the Fibonacci numbers where $y_0 = 0$, $y_1 = 1$ and $y_{n+1} = y_n + y_{n-1}$ for all $n \geq 2$. Let $z_n = y_{n-1}$ for $n \geq 1$. Then the Fibonacci sequence can be written as a first order recurrences system

$$y_{n+1} = y_n + z_n,$$
$$z_{n+1} = y_n$$

with initial conditions $y_1 = 1$ and $z_1 = 0$. By setting $X_n = \begin{pmatrix} y_n \\ z_n \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, one obtains

$$X_{n+1} = AX_n.$$

Diagonalize A and obtain a formula for the $(n+1)$-th Fibonacci number y_n.

5. (a) (10 points) Prove that similar square matrices have the same eigenvalues.

   (b) (10 points) Is $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ similar to $-A$? Prove your answer!

6. (20 points) Let $f : V \to W$ and $g : V \to W'$ be linear transformations such that $\ker g \subseteq \ker f$. Show that there exists a linear function $h : W' \to W$ such that $h \circ g = f$. (Hint. Consider extending a basis of $\ker g$ to a basis of V and remember that $\dim V = \dim(\ker g) + \dim(\operatorname{Im} g)$.)

End of Paper

國立中山大學 104 學年度碩士暨碩士專班招生考試試題

科目名稱：線性代數【應數系碩士班丙組】　　題號：424003

※本科目依簡章規定「不可以」使用計算機(問答申論題)　　共1頁第1頁

1 (10 pts) Find A^{-1} by Gauss-Jordan elimination with $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

2 (10 pts) A matrix $M \in R^{n\times n}$ is called skew-symmetric if $M^T = -M$. Prove the skew-symmetric matrices form a subspace of $R^{n\times n}$.

3 (10 pts) Let $A = \begin{bmatrix} 1 & 0 \\ a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & c & 0 \\ 0 & d & 1 \end{bmatrix}$, where a, b, c, d are non-zero real numbers.

(a) Find bases for the row and column spaces of A.

(b) Is A invertible?why?

4 (10 pts) Find the Jordan canonical form of matrix A, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

5 (10 pts) Prove that

$$\begin{vmatrix} \mathrm{O} & \mathrm{C} \\ \mathrm{A} & \mathrm{B} \end{vmatrix} = (-1)^m |\mathrm{A}||\mathrm{C}|,$$

where O is the $m \times m$ zero matrix and A, B and C are $m \times m$ matrices.

6 (50 pts) Prove or disprove.

(a) If A and B share the same column space, row space, null space, left null space then $A = B$.

(b) If rows of A are linearly dependent, so are columns.

(c) Let $A \in R^{n\times n}$ and $Ax = 0 \Rightarrow x = 0$, then $rank(A) = n$.

(d) Let W_1, W_2 are subspaces of V then $W_1 + W_2 = \{w_1 + w_2 | w_1 \in W_1, w_2 \in W_2\}$ is a subspace of V.

(e) Let A be a real $n \times n$ matrix, then A and its transpose A^t have the same minimal polynomial.