共十題，每題 10 分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. Positive integers a, b, and c are randomly and independently selected with replacement from the set $\{1, 2, 3, \ldots, 2010\}$. What is the probability that $abc + ab + a$ is divisible by 3?

2. Suppose we have 10 coins such that if the ith coin is flipped, heads will appear with probability $i/10$, $i = 1, 2, \ldots, 10$. When one of the coins is randomly selected and flipped, it show heads. What is the conditional probability that it was the fifth coin?

3. An insurance company writes a policy to the effect that an amount of money A must be paid if some event E occurs within a year. If the company estimates that E will occur within a year with probability p, what should it charge the customer in order that its expected profit will be 10 percent of A?

4. If X_1 and X_2 are independent exponential random variables with respective parameters λ_1 and λ_2, find the distribution of $Z = X_1/X_2$.

5. A prisoner is trapped in a cell containing 3 doors. The first door leads to a tunnel that returns him to his cell after 4 days travel. The third door leads to freedom after 1 day of travel. If it is assumed that the prisoner will always select doors 1, 2, and 3 with respective probabilities .5, .3, and .2. What is the expected number of days until the prisoner reaches freedom?

6. When the mean, median, and mode of the list

$$10, \ 2, \ 5, \ 2, \ 4, \ 2, \ x$$

are arranged in increasing order, they form a non-constant arithmetic progression (等差數列). What is the sum of all possible real values of x?

7. Suppose that the random variable X has the continuous uniform distribution

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Suppose that a random sample of $n = 12$ observations is selected from this distribution. What is the normal approximate probability distribution of $\overline{X} - 6$? Find the mean and variance of this quantity.

8. Let $X_1, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f(x) = \frac{1}{2\theta^3} x^2 e^{-x/\theta}, \quad 0 < x < \infty, \ 0 < \theta < \infty$$

Find the maximum likelihood estimator for θ.

9. A civil engineer is analyzing the compressive strength of concrete. Compressive strength is normally distributed with $\sigma^2 = 1000$ (psi)^2. A random sample of 12 specimens has a mean compressive strength of $\bar{x} = 3250$ psi. Construct a 95% two-sided confidence interval on mean compressive strength.

10. For a random sample $X_1, \ldots, X_n$ of Bernoulli(p) variables, it is desired to test

$$H_0 : p = .49 \quad \text{versus} \quad H_1 : p = .51.$$

Use the Central Limit Theorem to determine, approximately, the sample size needed so that the two probabilities of error are both about .01. Use $z_{.01} = 2.33$ and a test function that rejects H_0 if $\sum_{i=1}^{n} X_i$ is large.