

※ 考生請注意：本試題不可使用計算機。請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (20%) In an undamped mass-spring system, resonance occurs if the frequency of the driving force equals the natural frequency of the system and the model can be written as

$$y'' + \omega_0^2 y = K \sin \omega_0 t \text{ where } y(0) = y'(0) = 0$$

Solve above equations using Laplace transform.

Hint: You may use the convolution integral theorem :  $\mathcal{L}^{-1}(F(s)G(s)) = f * g$

2. (20%) Evaluate  $I = \int_0^\infty \frac{dx}{1+x^3}$  using contour integral in the complex plane.

3. (30%) Considering the following heat problem in dimensionless variables

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 6x, \quad 0 < x < 1, t > 0$$

$$\text{BC's: } u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0$$

$$u(x, 0) = T_0, \quad 0 < x < 1,$$

where  $T_0 > 0$  is a constant.

Derive the solution  $u(x, t)$  and find the steady-state (equilibrium) solution  $u_E$ .

4. (20%) (a) Complete the matrix  $\mathbf{A}$  (i.e. find  $a, b$ ) so that  $\mathbf{A}$  has eigenvectors  $\vec{x}_1 = (3, 1)$  and  $\vec{x}_2 = (2, 1)$ :

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ a & b \end{bmatrix}$$

(b) Find a different matrix  $\mathbf{B}$  with those same eigenvectors  $\vec{x}_1$  and  $\vec{x}_2$ , and with eigenvalues  $\lambda_1 = 1$  and  $\lambda_2 = 2$ . What is  $\mathbf{B}^{10}$  ?

5. (10%) Find the directional derivative of  $f(x, y, z) = 2x^2 + 3y^2 + z^2$  at point  $P : (2, 1, 3)$  in the direction of  $\vec{a} = (1, 0, 2)$ .