※ 注意：請於試卷內之「非選擇題作答區」依序作答，並應註明作答之部份及題號。

1. Solve the following linear ODE

   (a) $x^2y''+x(\frac{1}{2}+2x)y'+(x-\frac{1}{2})y=0$ (10%)

   (b) $ty''+(4t-2)y'-4y=0,\ y(0)=1$ (10%)

2. Solve the following linear ODE system (15%)

$$x_1'=x_1-10x_2+e^t$$
$$x_2'=-x_1+4x_2+\sin t$$

3. Please check the rank of the following matrix (10%)

   (a) $A=\begin{bmatrix} 1 & -1 & 4 & 2 \\ 0 & 1 & 3 & 2 \\ 4 & -2 & 15 & 8 \end{bmatrix}$

   (b) Consider a linear system Ax=b, please explain, in what rank conditions, this linear system has unique solution (5%)

4.

(a) $A=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \mathrm{con}\theta \end{bmatrix}$

When $\theta=22.5°$, find the eigenvalues and the corresponding eigenvectors? (5%)

When $\theta=60°$, find the eigenvalues and the corresponding eigenvectors? (5%).

(b) λ is an eigenvalue of $n\times n$ matrix A, and x is the corresponding eigenvector. Given a $n\times n$ nonsingular matrix P, prove that λ is the eigenvalues of matrix $(P^{-1}AP)$ and $(P^{-1}A)$ is the corresponding eigenvectors? Show the details of your work. (10%)

5. A periodic function $f_L(x)$ of period 2L is represented by a Fourier series:

$f_L(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L})$. Starting from "Fourier series", derive the method of "Fourier integral" by assuming $L\to\infty$. Show the details of your work. (10%)

6.

(a) Solve the following wave equation and show the details of your work. (10%)

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

$u(0,t)=0$ and $u(L,t)=0$ for all $t\geq 0$

$$u(x,0)=f(x)=\begin{cases} \frac{2x}{L}, & \text{if } 0<x<\frac{L}{2} \\ \frac{2(L-x)}{L}, & \text{if } \frac{L}{2}<x<L \end{cases}$$

$u_t(x,0)=0$ for $0\leq x\leq L$, where $L=8$ and $c=2$.

(b) Plot the graph showing the solution $u(x,t)$ at time, $t=0,1,2,3,$ and 4. (10%)