1. Consider the following ordinary differential equation: $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=3\sin t+\cos t$, with $y(0)=0$ and $y'(0)=0$.

(12%) (a) Find the homogeneous solution, the particular solution and the final solution.

(12%) (b) Find the solution using Laplace transform.

2. (10%) The function, $f(x)=\begin{cases}0, & -1/2<x<0\\ x, & 0<x<1/2\end{cases}$, is a section of a periodic function. Please expand this function into an appropriate Fourier series.

3. (6%) (a) Given A and B are both $n\times n$ matrices. Show that $A=A^{(s)}+A^{(a)}$, where $A^{(s)}$ and $A^{(a)}$ are respectively the symmetric and anti-symmetric matrices. Deduce $A^{(s)}$ and $A^{(a)}$. Compute $\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}B_{ji}$ in terms of $A^{(s)}$, $A^{(a)}$, $B^{(s)}$ and $B^{(a)}$.

(6%) (b) Given $A=\begin{bmatrix}-1 & 0 & -1\\ -1 & 2 & 0\\ 1 & 0 & 2\end{bmatrix}$. Find the triangular factorization $A = LU$, where $L=[L_{ij}]$ is a lower triangular matrix and $U=[U_{ij}]$ is an upper triangular matrix such that $L_{ij}=0$ when $i<j$ and $L_{ij}=1$ when $i=j$, and $U_{ij}=0$ when $i>j$.

(5%) (c) Let A and B be two $n\times n$ similar matrices, i.e., there exists a nonsingular $n\times n$ matrix P such that $A=PBP^{-1}$. Show that the eigenvalues of A and B are the same. Derive the relationship of the eigenvectors of A and B.

(7%) (d) Given a $n\times n$ non-symmetric matrix A with distinct real eigenvalues $\{\lambda_i\}$ and eigenvectors $\{\mathbf{u}_i\}$, where $i=1, 2, \cdots, n$. So $\{\mathbf{u}_i\}$ constitutes a basis in the n-dimensional space (i.e., any n-dimensional vector $\mathbf{x}$ can be expanded as a unique linear combination of basis $\{\mathbf{u}_i\}$, which is called the eigen-expansion of $\mathbf{x}$ in terms of the eigenvectors of matrix A). Show that the eigenvectors $\{\mathbf{v}_i\}$ of A^T is also a basis such that $\mathbf{u}_i\cdot\mathbf{v}_j=0$ if $i\neq j$. Then solve the algebraic equation $A\mathbf{x}=\mathbf{b}$ by eigen-expansion. Here $\mathbf{x}$ and $\mathbf{b}$ are $n\times 1$ vectors.

(4%) (e) A $n\times n$ symmetric matrix P is called to be positive definite if the associated quadratic form $\mathbf{x}^TP\mathbf{x}>0$ for all nonzero $n\times 1$ vector $\mathbf{x}$. Show that P possesses positive eigenvalues.

(5%) (f) Deduce the Green's 1st and 2nd identities:

$$\int_V u\nabla^2 v\,dV=\int_S u\frac{\partial v}{\partial n}dS-\int_V \nabla u\cdot\nabla v\,dV,\quad \int_V\left(u\nabla^2 v-v\nabla^2 u\right)dV=\int_S\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS,$$

where V is a three-dimensional domain, S is the surface of V, n is the coordinate along the normal $\mathbf{n}$ on S, and u and v are two scalar fields over V.

4. (18%) Solve the one-dimensional heat transfer (or diffusion) equation with source term,

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\pi^2\sin\pi x,$$ subject to $u(0,t) = 0$, $u(1,t) = 1$ and $u(x,0) = 0$.

5. Given a vector field $\mathbf{a}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2}$, where $(\hat{i},\hat{j},\hat{k})$ are unit vectors in the (x, y, z) coordinates system, calculate

(3%) (a) $\nabla\cdot\mathbf{a}$, (3%) (b) $\nabla\times\mathbf{a}$, (3%) (c) $\nabla(\nabla\cdot\mathbf{a})$,

(3%) (d) Directional derivative of $\nabla\cdot\mathbf{a}$ at the point (1,1,1) in the direction of $\hat{i}+2\hat{j}+3\hat{k}$,

(3%) (e) $\oint_C \mathbf{a}\cdot d\mathbf{l}$, where the integration is carried out along a closed circular path with radius 1 and center at (1,1,0) on the xy-plane.

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