共 9 大題。總分 100 分。

1. **(13 %)** Consider two propositions: $P: (A \vee B) \Rightarrow C$ and $Q: (C \Rightarrow B) \vee (\neg C \Rightarrow \neg A)$. Please draw a truth table (10 %) and decide the relationship between P and Q (3 %). You can use as many columns as you need.

2. **(12 %)** Show that the following statements about the integer x are equivalent: (a) $3x + 2$ is even, (b) $x + 5$ is odd, and (c) x^2 is even.

3. **(10 %)** Among the arrangements of the 26 different English letters: (a) How many ways can we permute to obtain "BRT" or "FAN"? (b) How many ways can we permute such that none of the patterns "USA" and "SAP" occur?

4. **(10 %)** Suppose that you have ten cards as shown below:

   1 1 2 2 3 3 4 4 5 5

   You shuffle them and deal them in a row and you might get:

   1 2 3 3 4 1 4 5 5 2

   What is the expected number of adjacent pairs with the same value? In the example, there are two adjacent pairs with the same value, the 3's and the 5's.

5. **(10 %)** Let A be an invertible matrix. Show that $(A^n)^{-1} = (A^{-1})^n$ whenever n is a positive integer.

6. **(10 %)** Prove that if $n \in \boldsymbol{Z}^+$ and $n \geq 2$, then $\prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$.

7. **(10 %)** Define the sequence of numbers as follows:

$$a_n = \begin{cases} 1 & \text{if } 0 \leq n \leq 3, \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} & \text{if } n \geq 4. \end{cases}$$

   Prove that $a_n \equiv 1 \pmod 3$ for all $n \geq 0$.

8. **(15 %)** Define $a_0 = 1,\ a_1 = 3,\ a_2 = 5$, and $a_n = a_{n-1} \cdot a_{n-2}^2 \cdot a_{n-3}^3$.
   (a) Write an iterative algorithm for finding the nth term of the sequence. (7 %)
   (b) Same as (a) but write a recursive algorithm instead. (5 %)
   (c) Which algorithm is more efficient? (3 %)

9. **(10 %)** Let $S = \{2,\ 16,\ 128,\ 1024,\ 8192,\ 65536\}$. If four numbers from S are arbitrary selected, show that the multiplication of any two numbers from the four selected numbers equals 131072.

試題隨卷繳回