注意事項：

1. 本試題共 6 題，配分共 100 分。
2. 請標明大題、子題編號作答，不必抄題。
3. 全部答案均須在答案卷之答案欄內作答，否則不予計分。

1. Assume X and Y are independent random variables with probability mass functions

$$P_X(x) = \begin{cases} 0.2, & x = 1 \\ 0.6, & x = 2 \\ 0.2, & x = 3 \end{cases} \qquad P_Y(y) = \begin{cases} 0.5, & y = 0 \\ 0.5, & y = 1 \end{cases}$$

Let $Z = X + Y$

(a) (10%) Find $\phi_Z(s)$, the moment generating function (MGF) of Z

(b) (5%) What is $E[Z^3]$

(c) (5%) What is the probability mass function $P_Z(z)$

2. Consider a binomial $(n, \frac{\alpha}{n})$ random variable X.

(a) (10%) As $n \to \infty$, show that the probability mass function (PMF) $P_X(x)$

　　　converges to the PMF of Poisson (α) random variable.

(b) (10%) As $n \to \infty$, use the central limit theorem to estimate the cumulative distribution

　　　function (CDF) of Z, assuming $Z = \dfrac{X - \alpha}{\sqrt{\alpha(1 - \frac{\alpha}{n})}}$

3. (10%) Suppose you have two coins, one biased, one fair, but your don't know which coin is which. Assume coin 1 is biased. It comes up heads with probability 3/4 while coin 2 will flip heads with probability 1/2. Suppose you pick a coin at random and flip it. Given that the outcome of the flip is a head, what is the probability that you picked the biased coin ?

4. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 6 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ -3 \\ 3 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 9 \\ -2 \\ -3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -5 \\ 1 \end{bmatrix}.$

The subspace W of $\mathbf{R}^4$ is spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

(a) (5%) Find a basis for the subspace W.

(b) (10%) Find an orthogonal basis for the subspace W.

(c) (5%) Find an orthonormal basis for the subspace W.

5. (15%) Let $A = \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix}$. Evaluate $A^{1/2}$.

6. Let $B = \begin{bmatrix} -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \\ 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \end{bmatrix}.$

Mark the following statements True or False. Justify each answer.

(a) (5%) B is an invertible matrix.

(b) (5%) The linear transformation $\mathbf{x} \mapsto B\mathbf{x}$ for any vector $\mathbf{x} \in \mathbf{R}^4$ is one-to-one.

(c) (5%) Dimension of the null space of B is n.