國立臺灣師範大學 101 學年度碩士班招生考試試題

科目：**線性代數與代數**　　　　　　適用系所：**數學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則不予計分。

Linear Algebra

Notations:

1. All vector spaces and matrices are over the real field $\mathbb{R}$.
2. For a linear transformation $T : V \to W$, $\ker(T) = \{v \in V \mid T(v) = 0\}$ and $\text{image}(T) = \{T(v) \in W \mid v \in V\}$.
3. $\mathfrak{M}_{m\times n}$ is the vector space of all $m \times n$ matrices. $I_m \in \mathfrak{M}_{m\times m}$ is the $m \times m$ identity matrix. For a given matrix $A \in \mathfrak{M}_{m\times n}$, $A^{\mathcal{T}} \in \mathfrak{M}_{n\times m}$ is the transpose of A.

Questions:

1. Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \in \mathfrak{M}_{2\times 2}$ and consider the standard basis of $\mathfrak{M}_{2\times 2}$

$$\mathbf{e}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{e}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $T : \mathfrak{M}_{2\times 2} \to \mathfrak{M}_{2\times 2}$ be a map given by

$$T(X) = A \cdot X - X^{\mathcal{T}} \cdot A.$$

(a) (10 %) Show that T is a linear transformation and give the matrix for T with respect to the standard basis.

(b) (10 %) Determine $\ker(T)$ and $\text{image}(T)$.

2. Let $A = \begin{bmatrix} -1 & -4 & 4 \\ 0 & 1 & 0 \\ -2 & -4 & 5 \end{bmatrix}$

(a) (8 %) Show that A is diagonalizable.

(b) (12 %) Find an invertible matrix U such that $A = UA^{\mathcal{T}}U^{-1}$.

3. (10 %) Given $A \in \mathfrak{M}_{m\times n}$ with $m \neq n$, prove that the set

$$\{B \in \mathfrak{M}_{n\times m} \mid AB = I_m\}$$

is either empty or an infinite set.

(背面尚有試題)

Algebra

4. Let G be a group and consider the map $\varphi : G \to G$ defined by $\varphi(a) = a^2$ for all $a \in G$. Prove that φ is a homomorphism if and only if G is abelian. (6%)

5. Let G be a group of order 54. Prove or disprove each of the following statements.

   (1) G has a subgroup of order 10. (5%)

   (2) G has a subgroup of order 9. (5%)

   (3) G has an element of order 9. (5%)

   (4) G is not simple. (5%)

6. Let R be the ring of all real value functions on the interval $[0, 3]$ and let $I = \{f \in R \mid f(2) = 0\}$.

   (1) Prove that I is an ideal of R. (6%)

   (2) Prove that I is a maximal ideal of R. (6%)

7. Let I be the principal ideal generated by $x^4 + x^2 + 1$ in $\mathbb{Q}[x]$, i.e., $I = (x^4 + x^2 + 1)$. Is $\mathbb{Q}[x]/I$ a field? Explain your answer. (6%)

8. Write 470 as a product of irreducible elements in $\mathbb{Z}[i]$. You don't need to explain your answer. (6%)

(試題結束)