國立臺灣師範大學 101 學年度碩士班招生考試試題

科目：**基礎數學**　　適用系所：**數學系**

注意：1.本試題共 1 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

1. (8 points) Find the tangent line of the following curve at the point (x_0, y_0),

$$x\sin(3y) = y\cos(2x), \quad (x_0, y_0) = (\frac{\pi}{4}, \frac{\pi}{3}).$$

2. (24 points) Evaluate the integrals

(a) $\int \frac{x+3}{2x^3 - 8x}\, dx,$

(b) $\int_0^2 x^2 e^{-x}\, dx,$

(c) $\int \frac{\sqrt{x^2 - 25}}{x^3}\, dx, \ x > 5.$

3. (8 points) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n (\ln n)^2}.$$

4. (10 points) Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

5. Consider the subspace

$$V = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + 3x_3 + 4x_4 = 0,\ x_2 - 2x_4 + 5x_5 = 0,\ x_1 - x_2 - x_3 + 3x_5 = 0\}$$

(a) (8pts) Find an orthogonal basis for V. What is the dimension of V?

(b) (5pts) Extend the basis you found in part (a) to a basis for $\mathbb{R}^5$.

(c) (5pts) Find the coordinate of $y = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \\ -1 \end{pmatrix}$ with respect to the basis in part (b).

6. Let V be the vector space consist of all polynomials with real coefficients having degree less than or equal to 3. Let T be the linear operator defined by

$$T(f(x)) = f(x) + f'(x) + (x^2 - x + 1)f''(x).$$

(a) (4pts) Let β be the standard ordered basis for V. Find the matrix representation of T with respect to β.

(b) (6pts) We know that T is invertible. Find the inverse function of T.

(c) (8pts) Determine whether T is diagonalizable. Give your reasons.

7. Let A be an $m \times n$ matrix. Prove that

(a) (6pts) $\text{rank}(A^T A) = \text{rank}(A)$, where A^T is the transpose of A.

(b) (8pts) Let B be an $n \times p$ matrix, then $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.