

# 國立臺灣師範大學 101 學年度碩士班招生考試試題

科目：高等微積分

適用系所：數學系

注意：1.本試題共 1 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則不予計分。

1. (10 分) Does the series  $\sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n^2}\right)$  converge or diverge? Justify your answer.
2. (10 分) Does the series  $f(x) = \sum_{n=0}^{\infty} x e^{-nx}$  converge uniformly on  $[0, \infty)$ ? Justify your answer.
3. (10 分) Prove that  $f(x) = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2$  is continuous on  $(-\infty, \infty)$ .
4. (10 分) (I) State the definition of path-connected set and connected set. (II) Give an example which is connected but not path-connected.
5. (10 分) Let  $a_0, a_1, a_2, a_3$  be four fixed real numbers such that  $a_0 + a_1 + a_2 + a_3 = 0$ . Prove that

$$\lim_{n \rightarrow \infty} (a_0 \sqrt{n} + a_1 \sqrt{n+1} + a_2 \sqrt{n+2} + a_3 \sqrt{n+3}) = 0.$$

6. (10 分) Show that for  $x > 0$ ,

$$x - \frac{x^2}{2} < \log(1+x) < x.$$

7. (10 分) Let  $P(x)$  be the polynomial of degree 5 defined as

$$P(x) = \frac{d^5}{dx^5} (x^2 - 1)^5.$$

Compute the definite integral  $\int_{-1}^1 [P(x)]^2 dx$ .

8. (10 分) Calculate the maximum value of the function  $f(x, y) = \frac{x^4 + 2x^3 y}{x^4 + y^4}$ , where  $(x, y) \in \mathbb{R}^2$ ,  $(x, y) \neq (0, 0)$ .

9. (10 分) Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and suppose there is a constant  $M$  such that for

$$x \in \mathbb{R}^2, \|f(x)\| \leq M^2 \|x\|^2. \text{ Prove that } f \text{ is differentiable at } x_0 = 0 \text{ and that } Df(x_0) = 0.$$

10. (10 分) Investigate whether the system

$$\begin{cases} u(x, y, z) = x + xyz \\ v(x, y, z) = y + xy \\ w(x, y, z) = z + 2x + 3z^2 \end{cases}$$

can be solved for  $x, y, z$  in terms of  $u, v, w$  near  $(0, 0, 0)$ .