國立臺灣師範大學101學年度碩士班招生考試試題

科目: **機率與統計**　　　　　　　　適用系所: **數學系**

注意:1. 本試卷共1頁,請依序在答案卷上作答,並標明題號,不必抄題。
2. 答案必須寫在指定作答區內, 否則不予計分。

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1. (20分) If X is a random variable such that $E(X) = 3$ and $\text{Var}(X) = 4$, determine a lower bound for the following probabilities.

   (a) $P(-2 < X < 8)$.

   (b) $P(0 < X < 10)$.

2. (20分) Let $w_s, s = 1, 2, \ldots$ be zero-mean independent random variables with variance σ^2 and $\mu_t = \sum_{s=1}^{t} w_s, t = 1, 2, \ldots$.

   (a) Find the expectation and variance of μ_t.

   (b) Find $\rho(\mu_t, \mu_r)$, the correlation of μ_t and μ_r, $t, r = 1, 2, \ldots$.

3. (20分) Let $U_1, U_2, \cdots, U_n$ be a random sample from a population with probability density function

$$f(u) = \frac{1}{\theta} u^{(1-\theta)/\theta}, 0 < u < 1$$

   where $\theta > 0$ is an unknown parameter. Let $V = -\frac{1}{\theta} \sum_{i=1}^{n} \log(U_i)$,

   (a) Find $E(V)$ and $\text{Var}(V)$.

   (b) Use the central limit theorem to obtain an approximate 95% confidence interval for θ based on V when n is large.

4. (20分) Consider the regression model

$$\begin{aligned} \mathbf{Y} &= \beta_1 \mathbf{1} + \beta_2 \mathbf{x}_c + \mathbf{e} \\ &= [\mathbf{1}\ \mathbf{x}_c] \binom{\beta_1}{\beta_2} + \mathbf{e} \\ &= \mathbf{X}\boldsymbol{\beta} + \mathbf{e} \\ &= \boldsymbol{\theta} + \mathbf{e} \end{aligned}$$

   where, $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)'$, $\mathbf{e} = (e_1, e_2, \cdots, e_n)'$, $\mathbf{1} = (1, 1, \cdots, 1)'$ is a $n \times 1$ vector, $\boldsymbol{\beta} = (\beta_1, \beta_2)'$, $\mathbf{x}_c = (x_1 - \bar{x}, x_2 - \bar{x}, \cdots, x_n - \bar{x})'$, $\mathbf{X} = [\mathbf{1}\ \mathbf{x}_c]$ and $\boldsymbol{\theta} = E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}$.

   (a) Find $\hat{\boldsymbol{\beta}} = (\hat{\beta}_1, \hat{\beta}_2)'$, the estimate of $\boldsymbol{\beta}$, using the least square method.

   (b) Let $\hat{\boldsymbol{\theta}} = \mathbf{X}\hat{\boldsymbol{\beta}}$ and $\hat{\mathbf{e}} = \mathbf{Y} - \hat{\boldsymbol{\theta}}$, show that the angle between the vectors $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{e}}$ is a right angle.

5. (20分) Let $X_1, X_2, \cdots, X_n \sim \text{Norm}(\mu, \sigma^2)$ and $Y_1, Y_2, \cdots, Y_n \sim \text{Norm}(\alpha\mu, \sigma^2)$ are two independent random samples, respectively, where $\sigma^2 > 0$ is known and both μ, and $\alpha(> 0)$ are unknown parameters.

   (a) Use Delta Method to find the limiting distribution of $\sqrt{n}(\frac{\overline{Y}}{\overline{X}} - \alpha)$.

   (b) Describe briefly how you will use (a) to test the null hypothesis $H_0 : \alpha = 1$ against the alternative hypothesis $H_1 : \alpha \neq 1$.