大同大學 101 學年度研究所碩士班入學考試試題

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Part I. Linear Algebra [50 Points]

I. (20%) True/False Problems:

Indicate true (◯) or false (×) for each below statement.
(2 points each, and with penalty of −1 point for each wrong answer.)

1. The empty set is a vector space over any field.
2. The vector space $\mathbb{R}^3$ has a basis containing the vector $(1, 2, 3)$.
3. If $V = \mathbf{span}\{v_1, \ldots, v_n\}$, then $dim(V) \leq n$.
4. If A is a non-singular 4×4 matrix, then $det(2A) = 2det(A)$.
5. Every matrix is row equivalent to a unique matrix in echelon form.
6. It is possible for the equation $\mathbf{Ax} = \mathbf{b}$ to have no solutions while at the same time for the equation $\mathbf{Ax} = \mathbf{0}$ to have infinitely many solutions.
7. If $\mathbf{A}$ is an $n \times n$ matrix with fewer than n distinct eigenvalues, then $\mathbf{A}$ is not diagonalizable.
8. All eigenvalues of a real symmetry matrix are real and distinct.
9. Let A and B be $n \times n$ matrices, with $AB = BA$. Then $A^3B = BA^3$.
10. If V is an inner product space and $u, v \in V$ satisfy $\langle u, v \rangle = 0$, then either $u = 0$ or $v = 0$.

II. (30%) Multiple-Choice Problems:

Each of the following problems has only one correct choice. Provide the correct choice for each of them.
(2 points each, without penalty on wrong choice.)

1. Let V be a five-dimensional vector space, and let S be a subset of V which is linearly independent. Then S
   (a) Must have at most five elements.
   (b) Must have exactly five elements.
   (c) Must have infinitely many elements.
   (d) Must consist of at least five elements.
   (e) Can have any number of elements (except zero).
   (f) Must be a basis for V.
   (g) Must span V.

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2. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation $T(x_1, x_2, x_3, x_4) \to (x_2, x_3, 0, 0)$. The null space (or kernel) $N(T)$ of T consists of all vectors of the form
   (a) $(0, 0, x_1, x_4)$ where x_1 and x_4 are real numbers
   (b) $(x_1, 0, 0, x_4)$ where x_1 and x_4 are real numbers
   (c) $(0, 0, x_3, x_4)$ where x_3 and x_4 are real numbers
   (d) $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$
   (e) $(x_2, x_3, 0, 0)$ where x_2 and x_3 are real numbers
   (f) $(0, x_2, x_3, 0)$ where x_2 and x_3 are real numbers
   (g) $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$

3. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if
   (a) The graph of T takes the form $y = mx + c$
   (b) There exists a $n \times m$ matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}^n$
   (c) One has $T(x + y) = T(x) + T(y)$ and $T(cx) = cT(x)$ for all vectors x, y and scalars c
   (d) One has $T(x + y) = T(x) + T(y)$ for all vectors x, y
   (e) T is one-to-one and onto
   (f) The image of T is a line
   (g) No condition required (all transformations are linear)

4. Let A be a 5×5 matrix with determinant 6. What is the determinant of A^{-1}?
   (a) 0
   (b) 6
   (c) 1
   (d) 1/6
   (e) 6/25
   (f) 25/6
   (g) Insufficient information to solve the question

5. Let A be an $n \times n$ matrix. Which of the following criteria will ensure that A is diagonalizable over the reals?
   (a) The characteristic polynomial of A has no repeated roots.
   (b) The characteristic polynomial of A splits over the reals.
   (c) A can be row reduced to a diagonal matrix.
   (d) The determinant of A is non-zero.
   (e) The rows of A are linearly independent.
   (f) A has n distinct real eigenvalues. (g) A is symmetry and invertible.

6. Let A be an invertible 5×5 matrix. Which of the following statements is false?
   (a) Every row of A must contain a leading 1.
   (b) There must exist a 5×5 matrix B, such that $AB = BA = I$.
   (c) The reduced row echelon form of A must be the identity matrix.
   (d) The rank of A must equal 5.
   (e) The linear transformation associated to A must be both one-to-one and onto.
   (f) The row-reduced echelon form of A must contain no free variables.
   (g) For every vector b in R^5, there must be exactly one solution to the equation $Ax = b$.

7. Let A be an invertible 5×5 matrix. Which of the following statements is false?
   (a) The image of A is $\mathbb{R}_5$.
   (b) The kernel of A is $\{0\}$.
   (c) There are five distinct eigenvalues.
   (d) The rows of A form a basis for $\mathbb{R}_5$.
   (e) The determinant of A is non-zero.
   (f) All the eigenvalues of A are non-zero.
   (g) The columns of A form a basis for $\mathbb{R}_5$.

8. 1.Let V be an inner product space, and let x and y be two vectors in V such that $||x|| = 3$ and $||y|| = 4$. What exactly can we say about $||x+y||$? (a) $||x+y|| = 5$.
   (b) $||x+y||$ is between 1 and 7 inclusive.
   (c) $||x+y||$ is less than or equal to 7.
   (d) $||x+y||$ is equal to 7.
   (e) $||x+y||$ is between 0 and 7 inclusive.
   (f) $||x+y||$ is less than or equal to 5.
   (g) $||x+y||$ is equal to 1 or 7.

9. Given $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 4 & 1 \\ 3 & 4 \end{bmatrix}$, $(AB)^{-1} =$?
   (a) $\begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -\frac{3}{2} \\ \frac{5}{2} & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$ (e) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -5 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} \frac{1}{2} & -\frac{5}{2} \\ 2 & 3 \end{bmatrix}$
   (g) $\begin{bmatrix} -\frac{1}{2} & \frac{5}{2} \\ 2 & -3 \end{bmatrix}$.

10. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$, then $\begin{vmatrix} 4g & a & d-2a \\ 4h & b & e-2b \\ 4i & c & f-2c \end{vmatrix} =$?
   (a) 6 (b) −12 (c) −24 (d) 12 (e) 24 (f) −6 (g) 8.

11. Given the matrices $A = \begin{bmatrix} 7 & 5 & 1 \\ 2 & 0 & 4 \\ 3 & 0 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & -1 \\ 1 & 2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 7 & 6 \\ 0 & 1 & 1 \\ 0 & 2 & -5 \end{bmatrix}$, which one of the following statements is true?
(a) only A is invertible
(b) only B is invertible
(c) only C is invertible
(d) A and B are both invertible
(e) B and C are both invertible
(f) A and C are both invertible
(g) none is invertible.

12. If A is a 3×3 matrix with an eigenvalue 2 whose associated eigenvector is $\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$, then $A^2 = A \cdot A$ has an eigenvalue λ whose associated eigenvector v is:
(a) $\lambda = 4$, $v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (b) $\lambda = 4$, $v = \begin{bmatrix} 3^2 \\ 6^2 \\ 3^2 \end{bmatrix}$ (c) $\lambda = 2$, $v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (d) $\lambda = 2$, $v = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$
(e) $\lambda = \sqrt{2}$, $v = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$ (f) $\lambda = \sqrt{2}$, $v = \begin{bmatrix} 3^2 \\ 6^2 \\ 3^2 \end{bmatrix}$ (g) none of the above.

13. Let W be a subspace of $\mathbb{R}^4$ with an orthogonal basis $\left\{ v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix}, v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ -3 \end{bmatrix} \right\}$.

We can write $u = \begin{bmatrix} 15 \\ -1 \\ -2 \\ 4 \end{bmatrix}$ as $u = u_1 + u_2$ where $u_1 \in W$, $u_2 \in W^{\perp}$. What it u_2?

(a) $u_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ (b) $u_2 = \begin{bmatrix} -8 \\ 0 \\ 7 \\ -3 \end{bmatrix}$ (c) $u_2 = \begin{bmatrix} 11 \\ 3 \\ -7 \\ 3 \end{bmatrix}$ (d) $u_2 = \begin{bmatrix} 5 \\ -3 \\ 7 \\ -3 \end{bmatrix}$ (e) $u_2 = \begin{bmatrix} 11 \\ -5 \\ 14 \\ -6 \end{bmatrix}$ (f) $u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
(g) none of the above.

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14. Compute $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{2007}$

(a) $\begin{bmatrix} 1 & 0 & 0 & -3^{2007} \\ 0 & 1 & 3^{2007} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 3^{2007} \\ 0 & 1 & -3^{2007} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & -6021 & 0 \\ 0 & 1 & 0 & 6021 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & 6021 \\ 0 & 1 & -6021 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 0 & -6021 \\ 0 & 1 & 6021 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(g) none of the above.

15. The matrix $A = \begin{bmatrix} 8 & -3 \\ 18 & 7 \end{bmatrix}$ is diagonalizable. Which of the following could be $P^{-1}AP$ for some invertible matrix P?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 4 & 0 \\ 0 & -5 \end{bmatrix}$ (g) $\begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$.

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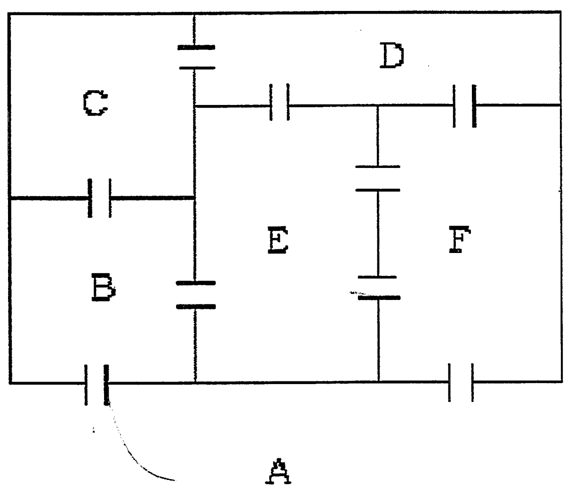
Part II. Discrete Math [50 Points]

True (T) or False (F) [20%]
2 points for each correct answer, -1 point for each wrong answer. Be careful.

1. $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$ are logically equivalent.
2. $\forall x(P(x) \to Q(x))$ and $\forall xP(x) \to \forall xQ(x)$ are logically equivalent.
3. For the domain of positive real numbers, both statements (a) $\exists x \forall y\ (x \leqq y^2)$ and (b) $\forall x \exists y\ (x \leqq y^2)$ are true.
4. The product of two irrational numbers is irrational.
5. If $x + y \geq 2$ then $x \geq 1$ or $y \geq 1$, where x and y are real numbers.
6. Suppose f: $\mathbf{N} \to \mathbf{N}$ has the rule $f(n) = 3n^2 + 1$. Then, f is 1-1.
7. Suppose f: $\mathbf{Z} \to \mathbf{Z}$ has the rule $f(n) = 3n + 1$. Then, f is onto.
8. A and B are two sets. If $A \cup B = A \cap B$, then $A = B$.
9. If $f(n)$ is $O(n^2)$ then $f(n)$ is also $O(n^3)$.
10. The picture shows the floor plan of an office. It is *possible* to plan a walk that passes through each doorway exactly once, starting and ending at A.



Multiple Choice [30%]
Each of the following questions has exactly one correct choice. 3 points for each correct choice, -1 point for each wrong choice. Be careful.

11. Suppose $A = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$. How many of these statements are TRUE?
$\varnothing \in A$,　$\varnothing \subseteq A$,　$\{\varnothing\} \in A$,　$\{\varnothing\} \subseteq A$,　$\varnothing \subseteq P(A)$,　$\{\varnothing\} \subseteq P(A)$.
a. 3
b. 4
c. 5
d. 6
e. none of the above

12. Which of the following describes the logical meaning of "John gets caught whenever he cheats"? (c for *caught*, h for *cheat*)
a. $c \to h$
b. $h \to c$
c. $c \wedge h$
d. $c \leftrightarrow h$
e. none of the above.

13. Which of the following is true?
a. If f_1 and f_2 are $O(g)$, then $f_1 f_2$ is $O(g)$.
b. If f is $O(g)$ then f is $O(g/2)$.
c. If f is $O(g)$ then g is $O(f)$.
d. If f is $O(g)$ then g is not $O(f)$.
e. none of the above.

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14. The relation R on $\boldsymbol{Z}$ where aRb means $|a| \geq |b|$. Determine whether the binary relation is: (a) reflexive, (b) symmetric, (c) antisymmetric, (d) transitive.

a. ad
b. abd
c. acd
d. abcd
e. none of the above

[for questions 15-17] Suppose $|A| = 3$ and $|B| = 6$.

15. Find the number of functions that can be defined from A to B, $f: A \to B$.

a. 3^6
b. 6^3
c. C(6,3)
d. P(6,3)
e. none of the above

16. Find the number of 1-1 functions that can be defined from A to B, $f: A \to B$.

a. 0
b. 6
c. 120
d. 20
e. none of the above

17. Find the number of binary relations that can be defined from A to B.

a. 18
b. 2^{18}
c. 6^3
d. 3^6
e. none of the above

18. How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

a. 3
b. 9
c. 42
d. 12
e. none of the above

19. How many terms are there in the expansion of $(3x - 2y + 4z)^{10}$?

a. 45
b. 55
c. 33
d. 66
e. none of the above

20. A zoo wants to set up natural habitats (棲息地) in which to exhibit the following animals, *rabbits, tigers, deers*(鹿), *lions,* and *owls*(貓頭鷹). Unfortunately, some animals will eat some of the others when given the opportunity. Therefore, they cannot be put in the same habitat. Use the graph theory ideas to determine the minimum number of different habitats needed.

	Tiger	Owl	Deer	Lion	Rabbit
Tiger			*	*	*
Owl					*
Deer	*			*	
Lion	*		*		*
Rabbit	*	*		*	

*: animals cannot be placed in the same habitat.

a. 1
b. 2
c. 3
d. 4
e. none of the above