機率部份 50分

1. (20%) The probability density function (*p.d.f.*) of a Chi-square random variable, X with $2n$ degrees of freedom is given by

$$f_X(x) = \begin{cases} \frac{1}{(n-1)!} x^{n-1} e^{-x} & , x \ge 0 \\ 0 & , \text{otherwise}, \end{cases}$$

where n is a positive integer.

(a) (10%) Find the expected value $\mathrm{E}\{e^{-\frac{X}{4}}\}$.
Hint: Use the fact that $\int_0^\infty t^{n-1} e^{-t} dt = (n-1)!$ for any positive integer n.

(b) (10%) Let Y be a Chi-square random variable with 2 degrees of freedom, and Y is independent of X. Find the probability $\mathrm{P}(Y \le \frac{X}{4})$.

2. (5%) Two cards are randomly chosen without replacement from an ordinary deck of 52 cards. Let B be the event that both cards are aces; let A be the event that at least one ace is chosen. Find the conditional probability $\mathrm{P}(B|A)$.

3. (15%) The lifetime a light bulb is an exponential random variable X with parameter λ i.e., the *p.d.f.* of the random X is defined as $f_X(x) = \lambda e^{-\lambda x}, \ x \ge 0$
(1). (10%) Describe and prove the memoryless property of the random variable X.
(2). (5%) Suppose 100 new light bulbs are installed at time t=0. Find the probability that all light bulbs are still working at time t=10. (Hint: Use the parameter λ to express the answer.)

4. (10%) Suppose that random variables X and Y are jointly Gaussian.
(1). (5%) Write down the joint *p.d.f.* of the random variables X and Y. (Hint: Use the mean, variances and correlation coefficient of X and Y i.e., $m_X, m_Y, \sigma_X, \sigma_Y, \rho_{XY}$ to express the joint *p.d.f.*)
(2). (5%) If X and Y are uncorrelated, are they independent? Prove your answer mathematically. (Hint: No credit will be given if there is no proof.)

線性代數部份 50分

5. (10%) Find a matrix S such that $S^2 = A$, if $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$.

6. (10%) Find the least squares solution of the linear system given by

$$\begin{aligned} x_1 \quad\quad\quad - x_3 &= 6 \\ 2x_1 + x_2 - 2x_3 &= 0 \\ x_1 + x_2 \quad\quad\quad &= 9 \\ x_1 + x_2 - x_3 &= 3 \end{aligned}$$

7. (10%) What conditions must b_1, b_2, and b_3 satisfy in order for the following system of equations to be consistent?

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 \quad\quad\quad + 8x_3 &= b_3 \end{aligned}$$

8. (10%) Let **u** and **v** be nonzero vectors in 2- or 3-space, and let $k = \|\mathbf{u}\|$ and $l = |\mathbf{v}|$. Show that the vector $\mathbf{w} = l\mathbf{u} + k\mathbf{v}$ bisects the angle between **u** and **v** (i.e., the angles between **u** and **w** and between **v** and **w** are equal).

9. (10%) Find the coordinate vector of **v** relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v} = (2,-1,3),\ \mathbf{v}_1 = (1,0,0),\ \mathbf{v}_2 = (2,2,0),\ \mathbf{v}_3 = (3,3,3)$$