## Graduate School Exam for Applied Mathematics 2012

1. (25 pts) (Laplace transform) Prove the Laplace transform of the function $f(t) = t^n$ is given by

$$L(t^n) = \frac{n!}{s^{n+1}},$$

where $n$ is 0 or a positive integer. The definition of the Laplace transform for a function $f(t)$ is given by

$$L\left(f(t)\right) = \int_0^\infty f(t) \times e^{-st}\, dt.$$

2. (25 pts) (Fourier series) An analytical function $f(x)$ is defined in $[-\pi, \pi]$ and can be represented by a Fourier series by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \times \cos nx + b_n \times \sin nx.$$

Show that $a_n$ and $b_n$ can be expressed as

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \times \cos nx\, dx$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \times \sin nx\, dx.$$

3. (25 pts) (Calculus) A Gaussian function can be defined as

$$G(x) = A \times e^{-x^2}.$$

Find the normalization constant $A$ such that

$$\int_{-\infty}^{\infty} G(x)\, dx = 1.$$

4. (25 pts) (Vector calculus) A vector field is given by

$$\vec{v} = yz\,\hat{i} + 3zx\,\hat{j} + z\,\hat{k}.$$

Find the **curl** for the vector field $\vec{v}$.