科目：線性代數　　　　系所組：數學

1. (40 Points) Prove each of the following statements.
   (a) If $\{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, then $\{v_1 + v_2, v_2 + v_3, \ldots, v_{n-1} + v_n, v_n\}$ is also a basis for V.
   (b) Suppose that S_1 and S_2 be subspaces for a vector space V such that $S_1 \subseteq S_2$. Then $\text{span}(S_1) \subseteq \text{span}(S_2)$.
   (c) Let T be an invertible linear operator on a finite-dimensional vector space V. Then λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}
   (d) Let T be a linear operator on an inner product space V with the adjoint T^*. If W is a T-invariant subspace of V, then $W^{\perp}$, the orthogonal complement of W, is T^*-invariant.
   (e) Let T be a linear operator on a complex inner product space V. If T is self-adjoint, then the inner product $\langle T(x), x\rangle$ is real for all x in V.

2. (20 Points) Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by

$$T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 + x_3 + x_4, x_2 - x_3 + x_4).$$

   (a) Let W be the null space of T. Find a basis for W.
   (b) Find a basis for the orthogonal complement of W.
   (c) Is T onto? Why?
   (d) If U is any linear transformation defined on $\mathbb{R}^2$ such that UT is onto, show that U is onto.

3. (16 Points) Let V be an inner product space. Denote $\langle\cdot,\cdot\rangle$ the inner product on V and define $\|x\| = \langle x, x\rangle^{1/2}$ for $x \in V$. If $S = \{v_1, v_2, \cdots, v_n\}$ is an orthonormal subset of V, show that for each $x \in V$ the following Bessel's inequality holds :

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i\rangle|^2$$

4. (24 Points) Let H be the vector space of continuous complex-valued functions defined on the interval $[0, 2\pi]$ endowed with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt, \qquad f, g \in H.$$

   (a) Let $f_n(t) = e^{int}$, where $0 \leq t \leq 2\pi$ and n is an integer. Show that $S = \{f_n : n \in \mathbf{Z}\}$ is an orthonormal subset of H.
   (b) Apply $f(t) = t$ in H for $x \in V$ in the Bessel's inequality and show that

$$\frac{\pi^2}{6} \geq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$