科目：線性代數　　　　　系所：應用數學系
考試時間：100 分鐘　　　身份別：一般生、在職生　　　是否使用計算機：否
　　　　　　　　　　　　本科原始成績：100 分

**Notations.**

$I_n$ : the identity matrix of size $n$.
$M_{n \times m}(\mathbb{R})$: the set of $n \times m$ real matrices.

1 (5) Find all scalars $s$, if any exist, such that $[1, 0, 1]$, $[2, s, 3]$, $[1, -s, 0]$ are independent.

2 Let $A, B \in M_{n \times n}(\mathbb{R})$
　a. (10) Prove that rank$(AB) \leq$ rank$(A)$.
　b. (5) Give an example where rank$(AB) <$ rank$(A)$.

3 (10) Let $u = [-1, 2]$ and $v = [3, -5]$ be in $\mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(u) = [-2, 1, 0]$ and $T(v) = [5, -7, 1]$. Find a formula for $T([x_1, x_2])$.

4 Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{bmatrix}.$$

　a. (10) Find the eigenvalues and eigenvectors of $A$.
　b. (5) Find the eigenvalues and eigenvectors of $3A^3 + 2A^2 + A + I_3$.

5 (10) Find a formula for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects vectors in the line $y = mx$, where $m \in \mathbb{R}$.

6 Let $W = \text{span}\{[1, 0, 1], [3, 1, 2]\}$.
　a. (5) Find an orthonormal basis for $W$.
　b. (10) Find the matrix that projects vectors in $\mathbb{R}^3$ on $W$.

7 (10) Find the least-squares fit to the data points, $[0, 0]$, $[1, 2]$, $[2, 3]$, $[3, 8]$, by a linear function $f(x) = r_0 + r_1 x$.

8 (10) Find the general solution of the linear differential equation

$$x' = x + y$$
$$y' = 3x - y$$

9 (10) Let $W_1$ and $W_2$ be subspaces of a vector space $V$. Prove that $W_1 \cap W_2$ is a subspace of $V$.