國立高雄大學 101 學年度研究所碩士班招生考試試題

科目：微積分
考試時間：100 分鐘

系所：應用數學系
身份別：一般生、在職生
本科原始成績：100 分

是否使用計算機：否

1. (10%) Determine whether the statement is True or False.
   (a) If $\sum a_n$ and $\sum b_n$ both converge, then $\sum a_n b_n$ converges.
   (b) The function $f(x) = e^x - 2$ and $g(x) = \ln(x+2)$ are inverses of each other.
   (c) If $f(x) = \ln x$, then $f(e^{n+1}) - f(e^n) = 1$ for any value of n.
   (d) If f' is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f'(x)\,dx = -f(0)$.
   (e) If $f(x) = g(x)$ for $x \neq c$ and $f(c) \neq g(c)$, then either f or g is not continuous at c.

2. (10%) Given an ε, δ proof for the limits

$$\lim_{x\to 1} |2 - 5x| = 3.$$

3. (10%) Evaluate the double integral.

$$(a)\ \int_0^1 \int_y^1 e^{y/x} dx dy; \qquad (b)\ \int_0^1 \int_0^{\sqrt{1-x^2}} \sin\sqrt{x^2+y^2} dx dy.$$

4. (10%) Assume that $u = u(x, y)$ is differentiable, and set $x = s + t, y = s - t$. Show that

$$\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 = \frac{\partial u}{\partial s}\frac{\partial u}{\partial t}.$$

5. (10%) Find the functions f and g such that

$$\lim_{x\to 1} f(x) = +\infty \quad \text{and} \quad \lim_{x\to 1} g(x) = +\infty$$

   but $\lim_{x\to 1} [f(x) - g(x)] = 0$.

6. (10%) Let $f(x) = \int_2^x (1+t^4)^{-1/2}\,dt$. Find the value of $(f^{-1})'(0)$.

7. (10%) Find the improper integral.

$$(a)\ \int_e^\infty \frac{1}{x(\ln x)^2} dx; \qquad (b)\ \int_1^\infty \frac{1}{(x-1)^2} dx.$$

8. (10%) Find the equation of the tangent plane to the paraboloid

$$x^2 + 2z^2 = y^2$$

   at the point $(1, 3, -2)$.

9. (10%) Test these series for (i) absolute convergence, (ii) conditional convergence (cannot only write the answer).

$$(a)\ \sum_{k=2}^\infty \frac{(-1)^k}{\ln k}; \qquad (b)\ \sum_{k=0}^\infty (-1)^k \frac{k^2}{2^k}.$$

10. (10%) Find the minimum value taken on by the function

$$f(x, y) = x^2 + (y-2)^2 \text{ on the hyperbola } x^2 - y^2 = 1.$$