國立東華大學招生考試試題

招生學年度	101	招生類別	碩士班
系所班別	應用數學系碩士班		
科目	線性代數		
注意事項	本考科禁止使用掌上型計算機		

1. (10分) Suppose you may use $D(f) = f'$, $(fg)' = f'g + fg'$, and $(e^x)' = e^x$. Let V be the subspace $\text{sp}(x^2e^x, xe^x, e^x)$ of the vector space of all differentiable functions mapping $\mathbb{R}$ into $\mathbb{R}$. Let $T : V \to V$ be the linear transformation of V into itself given by taking second derivatives, so $T = D^2$, and let $B = B' = (x^2e^x, xe^x, e^x)$. Find the matrix A representing T with respect to B and B'. (That is, $A = [T]_B^{B'}$.)

2. (10 points) Prove that for any $A, B \in M_n(\mathbb{C})$, AB and BA have the same eigenvalues.

3. (10 points in total, 2 points each) Let V be a finite dimensional inner product space over $\mathbb{C}$, A be a normal linear transformation on V, that is, $A^*A = AA^*$. Prove that

   (a) For any v in V, $||Av|| = ||A^*v||$.

   (b) $\ker A = \ker A^*$

   (c) $\forall \lambda \in \mathbb{C}$, $A - \lambda I$ is also normal.

   (d) If $Av = \lambda v$, then $A^*v = \overline{\lambda}v$.

   (e) If $\lambda_1 \neq \lambda_2$, $v_1 \neq 0 \neq v_2$, $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$, then $< v_1, v_2 >= 0$.

4. (10分) Prove that a $n \times n$ square matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

5. (10 points) Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 6 \end{bmatrix}$. Find out A^{1000}.

6. (10 points) Suppose you may use the fundamental theorem of Algebra: any nonconstant polynomial in $\mathbb{C}[x]$ has a root in $\mathbb{C}$. Prove Schur lemma: $\forall A \in M_n(\mathbb{C})$, $\exists\ U$: unitary, $\exists\ T$: upper triangular $\ni A = UTU^*$.

7. (10 points) Suppose you may use Schur lemma: $\forall A \in M_n(\mathbb{C})$, $\exists\ U$: unitary, $\exists\ T$: upper triangular $\ni A = UTU^*$. Prove Caylay-Hamilton theorem: If $p(\lambda)$ is the characteristic polynomial of $A \in M_n(\mathbb{C})$, then $p(A)$ is the zero matrix.

8. (10 points) Suppose you may use results in questions above. Let $A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 2 & 6 \end{bmatrix}$. Find out $A^{2012} - 14A^{2011} + 73A^{2010} - 168A^{2009} + 144A^{2008} + 10A + I$.

9. (10 points) Suppose you may use Schur lemma: $\forall A \in M_n(\mathbb{C})$, $\exists\ U$: unitary, $\exists\ T$: upper triangular $\ni A = UTU^*$. Prove that for $A \in M_n(\mathbb{C})$, if A is normal, then A is diagonalizable.

10. (10 points) Suppose you may use results in questions above. Prove that if $A \in M_n(\mathbb{R})$ and $A = A^T$, then $\exists U \in M_n(\mathbb{R}) \ni (U^TU = I = UU^T \wedge U^TAU \text{ is diagonal})$.