考試科目：計算機數學（含線性代數、離散數學）

系所名稱：資訊工程學系碩士班不分組

1. 答案以橫式由左至右書寫。2. 請依題號順序作答。

1. (30%) Let a transformation $T : R^3 \longrightarrow R^3$, is defined as

$$T(x_1, x_2, x_3) = (x_1 - 2x_2 + 2x_3, -3x_1 + 4x_2, -3x_1 + x_2 + 3x_3)$$

(a) Write down the standard matrix $[T]$ of the transformation. (3%)

(b) Find the image of the vector $\mathbf{v} = (-2, 1, 3)$ under this transformation. (3%)

(c) Show (證明) that the transformation is **linear** (線性的). (6%)

(d) Let the standard matrix $[T]$ be referred as matrix A, find its **eigenvalues** (特徵值) and corresponding **eigenvectors** (特徵向量). (6%)

(e) Is A **diagonalizable** (可對角線化)? If so, find a matrix P and its inverse P^{-1} that diagonalizes A, and determine the result of $P^{-1}AP$ (which is corresponding to the *increasing order* of eigenvalues, i.e., $\lambda_1 \le \lambda_2 \le \lambda_3$). (6%)

(f) Use the results of (e) to compute A^5. (6%)

2. (10%) For which values of a, will the linear system have no solutions? Exactly one solution? Infinitely many solutions?

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2. \end{aligned}$$

3. (10%) Let the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ with the coefficent matrix

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

(a) Find a basis for the **null space** (or kernel space) of A. (6%)

(b) Find the **rank** and the **nullity** of the matrix A. (4%)

(Hints:

1. The *null space* of a matrix A is the solution space of the linear system $A\mathbf{x} = \mathbf{0}$.

2. The rank of a matrix is the dimension of its column space (or row space).

3. The nullity of a matrix A is the dimension of its null space.)

4. (15%)(a)Use combinatorial argument to show $C_2^{2n} = 2C_2^n + n^2$

   (b)Use Pascal identity to prove $C_{n+1}^{2n} + C_n^{2n} = \frac{1}{2}C_{n+1}^{2n+2}$

5. (15%)Prove $\overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$ De Morgan's law

6. (20%)Solve the recurrence relation $a_n = 6a_{n-1} - 8a_{n-2} + 3^n$, $a_0 = 1$ $a_1 = 2$