科目：線性代數
考試時間：100 分鐘

系所：應用數學系
身份別：一般生、在職生
本科原始成績：100 分

是否使用計算機：否

Notations.

$M_{m\times n}(\mathbb{R})$: the set of all $m\times n$ matrices with entries in $\mathbb{R}$.

I_n: the identity matrix in $M_{n\times n}(\mathbb{R})$.

A^T: the transpose of matrix A.

$[T]_\beta^\gamma$: matrix representation of T relative to ordered bases β and γ.

1. (30 %) Determine "true" or "false" for the following statements. If true, prove it; if false, give a counterexample.
   (a) If $A\in M_{n\times n}(\mathbb{R})$ is invertible, then A is diagonalizable.
   (b) Two similar matrices have the same characteristic polynomial.
   (c) Let $A\in M_{3\times 3}(\mathbb{R})$ satisfy $A^3=A$, then A is diagonalizable.
   (d) For $A\in M_{n\times n}(\mathbb{R})$, the equation $AX-XA=I_n$ has a solution X in $M_{n\times n}(\mathbb{R})$.
   (e) For $A,B\in M_{n\times n}(\mathbb{R})$, AB and BA have the same eigenvalues.
2. Let T be the linear operator on $M_{n\times n}(\mathbb{R})$ defined by $T(A)=A^T$.
   (a) (10 %) Show that ± 1 are the only eigenvalues of T.
   (b) (5 %) Find an ordered basis β for $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta^\beta$ is a diagonal matrix.
3. Let $Q\in M_{3\times 3}(\mathbb{R})$ be an invertible matrix and let $S=\{A\in M_{3\times 3}(\mathbb{R}):\ AB=BA\}$, where
$$B=Q^{-1}\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}Q.$$
   (a) (5 %) Show that S is a subspace of $M_{3\times 3}(\mathbb{R})$.
   (b) (10 %) Find a basis for S.
4. Let A,B be in $M_{n\times n}(\mathbb{R})$.
   (a) (8 %) $\operatorname{rank}(A+B)\le\operatorname{rank}(A)+\operatorname{rank}(B)$.
   (b) (7 %) $\operatorname{rank}(A^TA)=\operatorname{rank}(AA^T)=\operatorname{rank}(A)$.

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5. (15 %) Let

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

Find an orthogonal matrix P such that $P^T AP$ is a diagonal matrix.

6. (10 %) Let W_1 and W_2 be subspaces of a vector space V over $\mathbb{R}$ such that $W_1 \cup W_2$ is also a subspace of V. Prove that $W_1 \subset W_2$ or $W_2 \subset W_1$.