第一部份：線性代數(50分)

1. There are three recursive functions

$$x_n = x_{n-1} + 0.5\, y_{n-1}$$
$$y_n = z_{n-1} + 0.5\, y_{n-1}$$
$$z_n = 0$$

where n is a positive integer. These relationships can be hold in the condition

$$x_n + y_n + z_n = 1 \ (n \text{ is a non-negative integer}).$$

Let $A^{(n)} = [x_n\, y_n\, z_n]^T$ be the n-th power of the matrix A.

(a) (5 %) Write these recursive equations in power form of matrix A with a coefficient matrix C.

(b) (5 %) Find the rank of C.

(c) (15 %) Find the eigenvalues and corresponding eigenvectors of C.

(d) (5%) Calculate the values of x_n, y_n, and z_n with the skill of "Diagonalization" as n approaches infinity.

2. If $p = \sum_{i=0}^{4} a_i x^i$ and $q = \sum_{i=0}^{4} b_i x^i$ are any vectors in P_4, then $\langle p, q \rangle = \sum_{i=0}^{4} a_i b_i$ is an inner product in P_4. Let V be the subspace of P_4 spanned by the vectors

$$p_1 = 2 + 2x - x^2 + x^4,$$
$$p_2 = -1 - x + 2x^2 - 3x^3 + x^4,$$
$$p_3 = 1 + x - 2x^2 - x^4,$$
$$p_4 = x^2 + x^3 + x^4.$$

(a) (5 %) Find the orthogonal complement of V.

(b) (5 %) Find a subset of $\{p_i | \ i = 1,2,3,4\}$ to be a basis of V.

3. Let $M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 4 & 4 \\ 1 & 2 & -2 & -3 \end{bmatrix}$

(a) (5 %) Show the determinant of M.

(b) (5 %) Calculate

$$\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & 4 & 4 \\ 1 & 2 & -2 & -3 \\ 4 & 4 & 4 & 4 \end{bmatrix} + \det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ -2 & -2 & 3 & 3 \\ 2 & 3 & -1 & -2 \end{bmatrix}.$$

第二部份：微分方程(50分)

Show all your work and write your answers clearly.

1. (10%) Solve the initial value problem

$$y' = -y + e^t y^2, \quad y(-1) = -1.$$

2. (10%) Solve the initial value problem

$$y'' + y' + y = 0, \quad y(0) = -2, \quad y'(0) = -2.$$

3. (10%) Solve the initial value problem

$$\mathbf{X}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} e^{2t} \\ -2t \end{pmatrix}, \quad \mathbf{X}(0) = \mathbf{0}.$$

4. (10%) Find the Frobenius series solutions of

$$xy'' + 2y' + xy = 0.$$

5. (10%) Solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$