系所組別：物理學系
考試科目：物理數學
考試日期：0223，節次：1

※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. Prove the following equalities. (20 points)

(a) $\nabla\cdot\left(\vec{a}\times\vec{b}\right)=\vec{b}\cdot\left(\nabla\times\vec{a}\right)-\vec{a}\cdot\left(\nabla\times\vec{b}\right)$

(b) $\nabla\times\left(\nabla\times\vec{a}\right)=\nabla\nabla\cdot\vec{a}-\nabla\cdot\nabla\vec{a}$

2. Solve the following equations for $y(x)$. (20 points)

(a) $(1+y)dx+(1-x)dy=0$

(b) $x\dfrac{dy}{dx}=2x+3y$

3. Consider the matrix $\begin{pmatrix}1&0&0\\0&1&1\\0&1&1\end{pmatrix}$. (20 points)

(a) Find the eigenvalues.

(b) Find the corresponding orthonormal eigenvectors.

(c) Compare the sum of the eigenvalues and the sum of the diagonal elements.

4. (15 points)

(a) Expand $\ln(1+x)$ as an infinite series for $-1<x\le 1$.

(b) Given the Riemann zeta function $\zeta(2)=\sum_{n=1}^{\infty}n^{-2}=\dfrac{\pi^2}{6}$, calculate $\int_0^1\dfrac{\ln(1+x)}{x}dx$.

5. Consider the periodic function $f(x)=\begin{cases}x, & 0<x<\pi\\ -x, & -\pi<x<0\end{cases}$ (15 points)

(a) Represent $f(x)$ by a Fourier series.

(b) Use the result of (a) to calculate $\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}$.

6. Use the generating function $e^{(x/2)(t-1/t)}=\sum_{n=-\infty}^{\infty}J_n(x)t^n$ to show that Bessel function $J_n(x)$ has odd or even parity according to whether *n* is odd or even, namely, $J_n(x)=(-1)^nJ_n(-x)$ (10 points)