考生請注意：本試題不可使用計算機。請於答案卷 (卡) 作答，於本試題紙上作答者，不予計分。

Note. Except the "True or False" questions, you need to provide complete argument to get full credit for any problem. If there is no argument, no credit will be given.

[16%] 1. True or False:

(a) Let A and D be $n \times n$ matrices. If D is diagonal, then $DA = AD$.

(b) Let A and B be $n \times n$ matrices. If B can be obtained by elementary row operations from A, then $\det(A) = \det(B)$.

(c) Let A and B be $n \times n$ matrices. If A and B are invertible, then $(A + B)$ is invertible.

(d) Let B be an $n \times n$ matrix. If B can be diagonalized, B must have n distinct eigenvalues.

(e) Any symmetric matrix is diagonalizable.

(f) The product of two symmetric matrices is symmetric.

(g) Similar matrices have the same characteristic polynomial.

(h) Any square matrix can be written as the product of a symmetric and skew-symmetric matrix.

[10%] 2. Consider the following three vectors in $\mathbb{R}^3$:

$$v_1 = (\lambda, 1, -\frac{1}{2})^t, v_2 = (1, \lambda, -\frac{1}{2})^t, v_3 = (1, -\frac{1}{2}, \lambda)^t.$$

If possible, find all values for $\lambda \in \mathbb{R}$ such that the three vectors are linearly dependent. If no value is possible, explain why not.

[10%] 3. Consider following subset of $\mathbb{R}^3$:

$$W = \left\{ \begin{pmatrix} a + 2b + 2c \\ -2b + c \\ a + 3c \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}.$$

Show that W is a subspace of $\mathbb{R}^3$ and find a basis for W.

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[12%] 4. Let $V = \{x \in \mathbb{R} \mid x > 0\}$. Consider the following operation on V:

$$x \oplus y = xy$$

for all $x, y \in V$. Is $(V, \oplus)$ a vector space over $\mathbb{R}$ if the scalar product is given by

$$\lambda \cdot x = x^{\lambda}$$

for all $x \in V$ and $\lambda \in \mathbb{R}$? If so, prove it. If not, explain.

[12%] 5. Let u, v, and w be three linearly independent vectors of a vector space over $\mathbb{R}$. Show that the vectors $u - 2v$, $v - 2w$, and $w - 2u$ are linearly independent.

[20%] 6. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

(a) Find the determinant of A.

(b) Find the characteristic polynomial and minimal polynomial of A.

(c) Find the inverse of A.

(d) Find the eigenvalues and corresponding eigenvectors of A.

[10%] 7. Consider the matrix

$$A = \begin{bmatrix} 12 & 7 & 1 \\ -2 & -4 & 0 \\ 0 & -8 & 2 \end{bmatrix}.$$

Is it possible to write $A = B + C$ where B is a symmetric matric and C a skew symmetric matrix? If so, find B and C; otherwise, explain.

[10%] 8. Let A be a square $n \times n$ matrix satisfying the following matrix equation $A^5 - 3A + I = 0$, where I is the $n \times n$ identity matrix. Show that A is invertible by finding the inverse of A.