系所組別： 電腦與通信工程研究所丙組
考試科目： 電磁數學

※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. Solve the following differential equations:

   (a) (10%) $x^2y' + y^2 - xyy' = 0$

   (b) (10%) $(1-x^2)y' - xy - xy^2 = 0$

2. (15%) Find the the general solution of the simultaneous equations

$$y + t\frac{dx}{dt} = 0$$
$$\frac{dy}{dt} - tx = 0$$

3. (15%) Solve the problem by using the Laplace transform method.

$$U_t = U_{xx}$$

I.C. (Initial condition) $U(x,0) = 3\sin 2\pi x \quad (0 \le x \le 1)$

B.C. (Boundary condition) $U(0,t) = U(1,t) = 0$

4. (25%) Mark each of the following statements True (T) or False (F). (Need not to give reasons.)

   (a) Suppose that A and B are two $n \times n$ matrices. The matrix AB is invertible if and only if both A and B are invertible.

   (b) For a 5×5 matrix A, if all the eigenvalues of A are non-zero, then the rank of A is 5.

   (c) For a square matrix A, if all the eigenvalues of A are zero, then the rank of A is 0.

   (d) Let T be a linear transformation from the vector space V to the vector space W. Then cT is also a linear transformation from V to W, where c is a constant scalar.

   (e) If both A and B are invertible $n \times n$ matrices, then $A + B$ is also an invertible matrix.

5. (10%) Suppose that a matrix A satisfies $A^2 = A$. Show the eigenvalues of A are either 1 or 0.

6. (15%) Suppose that we want to define an inner product in $\mathbb{C}^n$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^H A \mathbf{x}, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{C}^n,$$

where $\mathbf{y}^H = (\mathbf{y}^T)^*$ is the conjugate of $\mathbf{y}^T$. Explain why A must be positive-definite. ($\mathbb{C}$ denotes the set of all complex numbers.)