※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (30%) The jointly continuous random variables X and Y have a joint probability density function (pdf) that is uniform over the triangle with vertices at $(0,0)$, $(0,2)$, and $(2,0)$. Answer the following questions.
   (a) Find the marginal pdf of X.
   (b) Find the conditional pdf of X given Y.
   (c) What is the distribution of X when conditioned on a given $Y = y$ (with $0 < y < 2$).
   (d) Find $P(0.5 \leq X \leq 2 | Y = 0.5)$.
   (e) Find the conditional expectation $E[X|Y = y]$ where $0 < y < 2$.
   (f) Find the expectation $E[X]$.

2. (10%) The pair of jointly distributed random variables (X, Y) takes on the values $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$, each with probability $1/4$. Determine if X and Y are uncorrelated. Are X and Y independent? Justify your answers.

3. (10%) Let a and b be real numbers. For jointly distributed random variables X and Y, prove that

$$\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X, Y)$$

   where $\text{Var}(X)$ and $\text{Var}(Y)$ denote, respectively, the variance of X and that of Y, and $\text{Cov}(X, Y)$ denotes the covariance of X and Y.

   Note: The statement is a general result that is valid to both discrete and continuous random variables.

4. (25%) Mark each of the following statements True (T) or False (F). (Need not to give reasons.)
   (a) For a $n \times n$ matrix A, if all the eigenvalues of A are non-zero, then the rank of A is n.
   (b) For a square matrix A, if all the eigenvalues of A are zero, then the rank of A is 0.
   (c) If both A and B are invertible $n \times n$ matrices, then $A + B$ is also an invertible matrix.
   (d) Let T be a linear transformation from the vector space V to the vector space W. Then cT is also a linear transformation from V to W, where c is a constant scalar.
   (e) Suppose that A and B are two $n \times n$ matrices. The matrix AB is invertible if and only if both A and B are invertible.

5. (10%) Suppose that a matrix A satisfies $A^2 = A$. Show the eigenvalues of A are either 1 or 0.

6. (15%) Suppose that we want to define an inner product in $\mathbb{C}^n$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^H A\mathbf{x}, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{C}^n,$$

   where $\mathbf{y}^H = (\mathbf{y}^T)^*$ is the conjugate of $\mathbf{y}^T$. Explain why A must be positive-definite. ($\mathbb{C}$ denotes the set of all complex numbers.)