(10%) 1. Let $X_1,\dots,X_n$ $(n>2)$ be independent and identically distributed random variables. Find

$$E[X_2 \mid X_1+\dots+X_n=x]$$

(10%) 2. If $X_1, X_2,\dots, X_n$ are independent and identically distributed random variables having uniform distributions over $(0,1)$, find

(5%) (a) $E[\max(X_1,\dots,X_n)]$

(5%) (b) $E[\min(X_1,\dots,X_n)]$

(15%) 3. Let X and Y be independent $N(0,1)$ random variables, and define a new random variable Z by

$$Z=\begin{cases} X & if \quad XY>0 \\ -X & if \quad XY<0 \end{cases}.$$

(8%) (a) Show that Z has a normal distribution.

(7%) (b) Show that the joint distribution of Z and Y is not bivariate normal.

(10%) 4. Let $X_1,\dots,X_n$ be a random sample from a population with pdf

$$f_X(x)=\begin{cases} 1/\theta & if \quad 0<x<\theta \\ 0 & otherwise \end{cases}.$$

Let $X_{(1)}<\dots<X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent random variables.

(15%) 5. Let $X_1,\dots,X_n$ be a random sample from the pdf $f(x\mid\mu)=e^{-(x-\mu)}$, where $-\infty<\mu<x<\infty$.

(7%) (a) Show that $X_{(1)}=\min(X_1,\dots,X_n)$ is a complete sufficient statistic.

(8%) (b) Show that $X_{(1)}$ and S^2 (sample variance) are independent.

(15%) 6. Let $X_1,\ldots,X_n$ be a random sample from the pdf $f(x\mid\theta)$. Find a MLE of θ in each of the following cases.

(5%) (a) $f(x\mid\theta)=\theta^{-1}I_A(x)$, where $I_A(x)$ is an indicator function, $A=\{1,\ldots,\theta\}$, and θ is an integer between 1 and θ_0.

(5%) (b) $f(x\mid\theta)=\dfrac{\theta}{1-\theta}x^{(2\theta-1)/(1-\theta)}I_B(x)$, where $I_B(x)$ is an indicator function, $B=(0,1)$, and $\theta\in(1/2,1)$.

(5%) (c) $f(x\mid\theta)=\sigma^{-n}e^{-(x-\mu)/\sigma}I_C(x)$, where $I_C(x)$ is an indicator function, $C=(\mu,\infty)$, and $\theta=(\mu,\sigma)\in(-\infty,\infty)\times(0,\infty)$.

(15%) 7. Let $X_1,\ldots,X_n$ be a random sample from $N(\theta,1)$, $-\infty<\theta<\infty$.

(5%) (a) Find the UMVUE of θ^2.

(5%) (b) Find the UMVUE of θ^3.

(5%) (c) Find the UMVUE of θ^4.

(10%) 8. Let $Y_1<Y_2<\ldots<Y_5$ be the order statistics of a random sample of size $n=5$ from a distribution with pdf $f(x\mid\theta)=\frac{1}{2}e^{-|x-\theta|}$, $-\infty<x<\infty$, for all real θ. Find the likelihood ratio test for testing $H_0:\theta=\theta_0$ against $H_1:\theta\neq\theta_0$.