【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

每題都會依據作答情形給予部份分數，請勿因想法不夠完美就放棄該題；另一方面，要有完整的解釋並符合邏輯，才能得到全部分數。

1. (a) (6%) Prove that for integer $n \ge 0$, $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$.

   (b) (6%) Use a combinatorial argument to prove that for all integers n and k with $1 \le k \le n-1$, we have $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$.

   (c) (6%) Prove that for positive integers n and k,

   $$\binom{n+1}{k+1}=\binom{0}{k}+\binom{1}{k}+\cdots+\binom{n-1}{k}+\binom{n}{k}.$$

   (d) (7%) Let n be a positive integer and suppose n is even. Prove that the sequence $\binom{n}{0},\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{n}$ satisfies $\binom{n}{0}<\binom{n}{1}<\cdots<\binom{n}{n/2}$, $\binom{n}{n/2}>\cdots>\binom{n}{n-1}>\binom{n}{n}$.

2. (a) (5%) Prove that if $n+1$ objects are put into n boxes, then at least one box contains ≥ 2 objects.

   (b) (5%) Let $q_1, q_2, \ldots, q_n$ be positive integers. Prove that if $q_1+q_2+\cdots+q_n-n+1$ objects are put into n boxes, then either the 1st box contains at least q_1 objects or the 2nd box contains at least q_2 objects, ..., or the n-th box contains at least q_n objects.

   (c) (10%) Let m and n be relatively prime positive integers, and let a and b be integers where $0 \le a \le m-1$ and $0 \le b \le n-1$. Prove that there exists a positive integer k such that the remainder when k is divided by m is a, and the remainder when k is divided by n is b.

3. (a) (8%) We call a combination containing n objects an n-combination. Determine the number of 10-combinations of $\{5\cdot a, 4\cdot b, 3\cdot c, 4\cdot d\}$, i.e., of $\{a,a,a,a,a,b,b,b,b,c,c,c,d,d,d,d\}$.

(b) (8%) How many permutations of the 9 letters W, H, O, I, S, M, A, R, Y are there such that none of the words WHO, IS, MARY occurs as consecutive letters? For example, WHIOMASRY is counted but WHIOMARYS is not counted.

(c) (10%) A derangement of $\{1,2,\dots,n\}$ is a permutation $i_1 i_2 \cdots i_n$ of $\{1,2,\dots,n\}$ such that $i_1 \neq 1, i_2 \neq 2, \cdots, i_n \neq n$. Let D_n denotes the number of derangements of $\{1,2,\dots,n\}$. Give the values of D_2 and D_3 and prove that $D_n = n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}\right)$.

4. We consider simple and undirected graphs.

(a) (8%) Prove that a graph of order n with at least $\binom{n-1}{2}+1$ edges is always connected.

(b) (7%) Let $d_1, d_2, \dots, d_n$ be the degree sequence of a tree of order $n \geq 2$ (note: $d_1 \leq d_2 \leq \cdots \leq d_n$). Prove that $d_1 = d_2 = 1$.

(c) (7%) A graph G of order n is said to satisfy the Ore property if for all pairs of distinct vertices x and y that are not adjacent, $d(x)+d(y) \geq n$, where $d(v)$ denotes the degree of vertex v. It has been proved that

(*) A graph G of order $n \geq 3$ that satisfies the Ore property has a Hamiltonian cycle.

Use (*) to prove that a graph of order $n \geq 3$, in which each vertex has degree at least $n/2$, has a Hamiltonian cycle.

(d) (7%) Each of the following two graphs G and G' has 6 vertices and 9 edges. Prove or disprove that they are isomorphic.

G 　　　　 G'