國立交通大學 103 學年度碩士班考試入學試題

科目:離散數學(4041)

考試日期:103年2月14日 第3節

系所班別:應用數學系

組別: 應數系乙組

第 1 頁,共2 頁

【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符!!

每題都會依據作答情形給予部份分數,請勿因想法不夠完美就放棄該題;另一方面,要有完整的解釋並符合邏輯,才能得到全部分數。

- 1. (a) (6%) Prove that for integer $n \ge 0$, $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$.
 - (b) (6%) Use a <u>combinatorial argument</u> to prove that for all integers n and k with $1 \le k \le n-1$, we have $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
 - (c) (6%) Prove that for positive integers n and k,

$$\binom{n+1}{k+1} = \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k}.$$

(d) (7%) Let n be a positive integer and suppose n is even. Prove that the sequence

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n} \text{ satisfies } \binom{n}{0} < \binom{n}{1} < \dots < \binom{n}{n/2}, \quad \binom{n}{n/2} > \dots > \binom{n}{n-1} > \binom{n}{n}.$$

- 2. (a) (5%) Prove that if n+1 objects are put into n boxes, then at least one box contains ≥ 2 objects.
 - (b) (5%) Let $q_1, q_2, ..., q_n$ be positive integers. Prove that if $q_1 + q_2 + ... + q_n n + 1$ objects are put into n boxes, then either the 1st box contains at least q_1 objects or the 2nd box contains at least q_2 objects, ..., or the n-th box contains at least q_n objects.
 - (c) (10%) Let m and n be relatively prime positive integers, and let a and b be integers where $0 \le a \le m-1$ and $0 \le b \le n-1$. Prove that there exists a positive integer k such that the remainder when k is divided by m is a, and the remainder when k is divided by n is b.
- 3. (a) (8%) We call a combination containing n objects an n-combination. Determine the number of 10-combinations of $\{5 \cdot a, 4 \cdot b, 3 \cdot c, 4 \cdot d\}$, i.e., of $\{a, a, a, a, a, b, b, b, b, c, c, c, d, d, d, d\}$.

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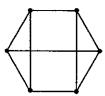
第 2 頁,共2 頁

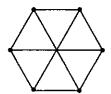
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(b) (8%) How many permutations of the 9 letters W, H, O, I, S, M, A, R, Y are there such that none of the words WHO, IS, MARY occurs as consecutive letters? For example, WHIOMASRY is counted but WHIOMARYS is not counted.

- (c) (10%) A derangement of $\{1,2,...,n\}$ is a permutation $i_1i_2\cdots i_n$ of $\{1,2,...,n\}$ such that $i_1 \neq 1, i_2 \neq 2, \cdots, i_n \neq n$. Let D_n denotes the number of derangements of $\{1,2,...,n\}$. Give the values of D_2 and D_3 and prove that $D_n = n! \left(1 \frac{1}{1!} + \frac{1}{2!} \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$.
- 4. We consider simple and undirected graphs.
 - (a) (8%) Prove that a graph of order n with at least $\binom{n-1}{2}+1$ edges is always connected.
 - (b) (7%) Let $d_1, d_2, ..., d_n$ be the degree sequence of a tree of order $n \ge 2$ (note: $d_1 \le d_2 \le ... \le d_n$). Prove that $d_1 = d_2 = 1$.
 - (c) (7%) A graph G of order n is said to satisfy the Ore property if for all pairs of distinct vertices x and y that are not adjacent, $d(x)+d(y) \ge n$, where d(v) denotes the degree of vertex v. It has been proved that
 - (*) A graph G of order $n \ge 3$ that satisfies the Ore property has a Hamiltonian cycle. Use (*) to prove that a graph of order $n \ge 3$, in which each vertex has degree at least n/2, has a Hamiltonian cycle.
 - (d) (7%) Each of the following two graphs G and G' has 6 vertices and 9 edges. Prove or disprove that they are isomorphic.

G





G'